EUCLIDEAN PARTITIONS THAT OPTIMIZE AN ORNSTEIN-UHLENBECK QUADRATIC FORM

STEVEN HEILMAN

Abstract. The Standard Simplex Conjecture of Isaksson and Mossel [9] asks for the partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ into k pieces of equal Gaussian measure that minimizes a generalized Gaussian perimeter. These authors guessed the best partition for this problem and proved some applications of their conjecture. For example, the Standard Simplex Conjecture implies the Plurality is Stablest Conjecture. For $0<\rho<1$ we maximize the following quantity:

$$\sum_{i=1}^{k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{A_i}(x)1_{A_i}(x\rho+y\sqrt{1-\rho^2})e^{-(x_1^2+\cdots+x_n^2)/2}e^{-(y_1^2+\cdots+y_n^2)/2}dxdy.$$

For $k=3, n\geq 2$ and $0<\rho<\rho_0(k,n)$, we prove the Standard Simplex Conjecture. This conjecture has applications to theoretical computer science [9, 10, 15] and to geometric multi-bubble problems.

1. Introduction

The Standard Simplex Conjecture [9] asks for the partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ into $k\leq n+1$ sets of equal Gaussian measure that minimizes a generalized Gaussian perimeter. This Conjecture generalizes a seminal result of Borell, [3, 15], which says that the two regions of fixed Gaussian measures $0<a<1$ and $1-a$ and of minimal generalized Gaussian perimeter must be separated by a hyperplane. We prove a specific case of this conjecture for $k=3$. We first discuss consequences of the full conjecture, and we then state the conjecture precisely. The Standard Simplex Conjecture appears to be first stated explicitly in [9]. If true, this conjecture implies:

- Optimal hardness results for approximating the MAX-k-CUT problem [9][Theorem 1.13], a generalization of the MAX-CUT problem. (These hardness results are optimal, assuming the Unique Games Conjecture).
- The Plurality is Stablest Conjecture [10],[9][Theorem 1.10], an extension of the Majority is Stablest Conjecture [15] asserting that: the most noise-stable way to determine the winner of an election between k candidates is to take the plurality. (This result assumes that no one person has too much influence over the election's outcome).
- The solution of a multi-bubble problem in Gaussian space [5, 9, 14]: in $\mathbb{R}^n$, minimize the total Gaussian perimeter of $k\leq n+1$ sets of Gaussian measure $1/k$.

For a survey of the motivation for problems such as this one, see [13]. For applications and related work, see [6, 16, 11, 2, 12, 9, 4, 8]. We now describe the Standard Simplex Conjecture. Let $\rho\in[-1,1]$, $n\geq 1$, $n\in\mathbb{Z}$, let $f\colon\mathbb{R}^n\to\mathbb{R}$ be bounded and measurable, and

This research is supported by NSF Graduate Research Fellowship DGE-0813964. Part of this work was carried out while visiting the Quantitative Geometry program at MSRI.

define $d\gamma_n(y) := e^{-(y_1^2+\cdots+y_n^2)/2}dy/(2\pi)^{n/2}$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$, define

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y). \tag{1}$$

Definition 1.1. Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$ be measurable. We say that $\{A_i\}_{i=1}^k$ is a **partition** of $\mathbb{R}^n$ if $\cup_{i=1}^k A_i = \mathbb{R}^n$, and $\gamma_n(A_i \cap A_j) = 0$ for $i \neq j$, $i, j \in \{1, \ldots, k\}$. Let $\{z_i\}_{i=1}^k$ be the vertices of a regular simplex centered at the origin of $\mathbb{R}^n$. Define $A_i := \{x \in \mathbb{R}^n : \langle x, z_i\rangle = \max_{j\in\{1,\ldots,k\}}\langle x, z_j\rangle\}$. We call $\{A_i\}_{i=1}^k$ a **regular simplicial conical partition**.

Conjecture 1 (**Standard Simplex Conjecture**, [9]). *Let $n \geq 2$, let $\rho \in [-1, 1]$, and let $3 \leq k \leq n+1$. Let $\{A_i\}_{i=1}^k$ be a partition of $\mathbb{R}^n$.*

(a) *If $\rho \in (0, 1]$, and if $\gamma_n(A_i) = 1/k$, $\forall\, i \in \{1, \ldots, k\}$, then among all such partitions of $\mathbb{R}^n$, the quantity*

$$J := \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i}(x) T_\rho(1_{A_i})(x) d\gamma_n(x) \tag{2}$$

is maximized for the regular simplicial conical partition.

(b) *If $\rho \in [-1, 0)$ (with no restriction on the measures of the sets A_i, $i \in \{1, \ldots, k\}$), then among all partitions of $\mathbb{R}^n$, the quantity J is minimized for the regular simplicial conical partition.*

The following theorem is our main result.

Theorem 1.2 (**Main Theorem**). *Fix $n \geq 2$, $k = 3$. There exists $\rho_0 = \rho_0(n, k) > 0$ such that Conjecture 1 holds for $\rho \in (0, \rho_0)$.*

Theorem 1.2 seems to have no direct relation to Gaussian bubble problems [5]. Also, [9][Lemma A.4,Theorem A.6] shows that Theorem 1.2 seems to give no new information about the MAX-k-CUT problem.

To see that our formulation of Conjecture 1 is equivalent to that of [9], let $A \subseteq \mathbb{R}^n$ and note that

$$\begin{aligned}\int 1_A T_\rho 1_A d\gamma_n &= \int 1_A(x) \int 1_A(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y) d\gamma_n(x) \\ &= \iint 1_A(x) 1_A(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y) d\gamma_n(x) = \mathbb{P}((X, Y) \in A \times A).\end{aligned}$$

Here $X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n)$ are jointly normal standard n-dimensional Gaussian random variables such that the covariances satisfy $\mathbb{E}(X_i Y_j) = \rho \cdot 1_{\{i=j\}}$, $i, j \in \{1, \ldots, n\}$.

We now describe the complexity theoretic notions referenced above, along with some history of Conjecture 1. Hardness results for the MAX-k-CUT [6] and Γ-MAX-2LIN(k) problems led to the creation of the Plurality is Stablest Conjecture [10]. In the MAX-k-CUT problem, we try to partition a given graph into k pieces of maximal boundary size. And in the Γ-MAX-2LIN(k) problem, we try to satisfy a maximum number of given two-term linear equations mod k. A special case of the Plurality is Stablest Conjecture, the Majority is Stablest Conjecture, was solved in [15]. In [15], one key insight was to pass between the discrete and continuous versions of the problem at hand. The discrete Fourier-analytic version of the problem, the Majority is Stablest problem, was posed on the hypercube $\{-1, 1\}^m$. The other version of the problem, a continuous isoperimetric result, was posed on

Gaussian space $(\mathbb{R}^n, d\gamma_n)$. The latter problem had been solved by Borell [3], and the discrete problem was desired. For large m, a nonlinear generalization of the central limit theorem was used to relate the problem on the hypercube to the corresponding problem on Gaussian space. This theorem is referred to as an invariance principle.

Influenced by this passage between discrete and continuous problems, [9] found that the discrete Fourier-analytic Plurality is Stablest Conjecture could be solved by its continuous isoperimetric analogue, the Standard Simplex Conjecture. However, unlike the previous work [15], in [9] no one had solved the continuous version of the desired problem. We are therefore partially motivated to solve this continuous problem, Conjecture 1, to attempt to complete the picture set out by this sequence of works [3, 10, 15, 9]. The four items listed below then summarize the original complexity theoretic motivation of Conjecture 1.

Definition 1.3 (**MAX-k-CUT**)**.** Let $k \in \mathbb{N}$, $k \geq 2$. We define the weighted MAX-k-CUT problem. In this problem, we are given a finite graph, defined by a vertex set V and an edge set $E \subseteq V \times V$. We are also given a weight function $w\colon E \to [0,1]$. A k-cut is a function $c\colon V \to \{1,\ldots,k\}$. The goal of the MAX-k-CUT problem is to find the following quantity:

$$\max_{c\colon V \to \{1,\ldots,k\}} \sum_{\substack{(i,j)\in E\colon\\ c(i)\neq c(j)}} w(i,j).$$

Definition 1.4 (**Γ-MAX-2LIN(k)**)**.** Let $k \in \mathbb{N}$, $k \geq 2$. We define the Γ-MAX-2LIN(k) problem. In this problem, we are given $m \in \mathbb{N}$ and $2m$ variables $x_i \in \mathbb{Z}/k\mathbb{Z}$, $i \in \{1,\ldots,2m\}$. We are also given a set $E \subseteq \{1,\ldots,2m\}\times\{1,\ldots,2m\}$ of cardinality m. An element $(i,j) \in E$ corresponds to one of m linear equations of the form $x_i - x_j = c_{ij} (\mathrm{mod}\, k)$, $i,j \in \{1,\ldots,2m\}$, $c_{ij} \in \mathbb{Z}/k\mathbb{Z}$. We are also given a weight function $w\colon E \to [0,1]$. The goal of the Γ-MAX-2LIN(k) problem is to find the following quantity:

$$\max_{(x_1,\ldots,x_{2m})\in(\mathbb{Z}/k\mathbb{Z})^{2m}} \sum_{\substack{(i,j)\in E\colon\\ x_i - x_j = c_{ij} (\mathrm{mod}\, k)}} w(i,j). \tag{3}$$

Definition 1.5 (**Unique Games Conjecture**, [10])**.** For every $\varepsilon \in (0,1)$, there exists a prime number $p(\varepsilon)$ such that no polynomial time algorithm can distinguish between the following two cases, for instances of Γ-MAX-2LIN($p(\varepsilon)$) with $w = 1$:

- (i) (3) is larger than $(1-\varepsilon)m$, or
- (ii) (3) is smaller than εm.

Theorem 1.6. (**Optimal Approximation for MAX-k-CUT**, [9][Theorem 1.13],[6])**.** *Let $k \in \mathbb{N}$, $k \geq 2$. Let $\{A_i\}_{i=1}^k \subseteq \mathbb{R}^{k-1}$ be a regular simplicial conical partition. Define*

$$\alpha_k := \inf_{-\frac{1}{k-1}\leq\rho\leq 1} \frac{k - k^2 \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} T_\rho 1_{A_i} d\gamma_n}{(k-1)(1-\rho)} = \inf_{-\frac{1}{k-1}\leq\rho\leq 0} \frac{k - k^2 \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} T_\rho 1_{A_i} d\gamma_n}{(k-1)(1-\rho)}.$$

Assume Conjecture 1 and the Unique Games Conjecture. Then, for any $\varepsilon > 0$, there exists a polynomial time algorithm that approximates MAX-k-CUT within a multiplicative factor $\alpha_k - \varepsilon$, and it is NP-hard to approximate MAX-k-CUT within a multiplicative factor of $\alpha_k + \varepsilon$.

1.1. **Plurality is Stablest.** We now briefly describe the Plurality is Stablest Conjecture. This Conjecture seems to first appear in [10]. The work [10] emphasizes the applications of this conjecture to MAX-k-CUT and to MAX-2LIN(k).

Let $n \geq 2, k \geq 3$ Let $(W_1, \ldots, W_k)$ be an orthonormal basis for the space of functions $\{g\colon \{1,\ldots,k\} \to [0,1]\}$ equipped with the inner product $\langle g, h\rangle_k := \frac{1}{k}\sum_{\sigma\in\{1,\ldots,k\}} g(\sigma)h(\sigma)$. Assume that $W_1 = 1$. By orthonormality, there exist $\widehat{g}(\sigma) \in \mathbb{R}$, $\sigma \in \{1,\ldots,k\}$, such that the following expression holds: $g = \sum_{\sigma\in\{1,\ldots,k\}} \widehat{g}(\sigma)W_\sigma$. Define

$$\Delta_k := \{(x_1,\ldots,x_k) \in \mathbb{R}^k \colon \forall 1 \leq i \leq k, 0 \leq x_i \leq 1, \sum_{i=1}^{k} x_i = 1\}.$$

Let $f\colon \{1,\ldots,k\}^n \to \Delta_k$, $f = (f_1,\ldots,f_k)$, $f_i\colon \{1,\ldots,k\}^n \to [0,1]$, $i \in \{1,\ldots,k\}$. Let $\sigma = (\sigma_1,\ldots,\sigma_n) \in \{1,\ldots,k\}^n$. Define $W_\sigma := \prod_{i=1}^{n} W_{\sigma_i}$, and let $|\sigma| := |\{i \in \{1,\ldots,n\}\colon \sigma_i \neq 0\}|$. Then there exists $\widehat{f_i}(\sigma) \in \mathbb{R}$ such that $f_i = \sum_{\sigma\in\{1,\ldots,k\}^n} \widehat{f_i}(\sigma)W_\sigma$, $i \in \{1,\ldots,k\}$.

For $\rho \in [-1,1]$ and $i \in \{1,\ldots,k\}$, define

$$T_\rho f_i := \sum_{\sigma\in\{1,\ldots,k\}^n} \rho^{|\sigma|}\widehat{f_i}(\sigma)W_\sigma, \quad T_\rho f := (T_\rho f_1,\ldots,T_\rho f_k) \in \mathbb{R}^k.$$

Let $m \geq 2$, $k \geq 3$. Let $e_j = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^k$ be the j^{th} unit coordinate vector. Let $\sigma \in \{1,\ldots,k\}^n$. Define $\mathrm{PLUR}_{m,k}\colon \{1,\ldots,k\}^m \to \Delta_k$ such that

$$\mathrm{PLUR}_{m,k}(\sigma) := \begin{cases} e_j & \text{, if } \left|\{i \in \{1,\ldots,m\}\colon \sigma_i = j\}\right| > \left|\{i \in \{1,\ldots,m\}\colon \sigma_i = r\}\right|, \\ & \qquad \forall\, r \in \{1,\ldots,k\} \\ \frac{1}{k}\sum_{i=1}^{k} e_i & \text{, otherwise} \end{cases}$$

Conjecture 2 (**Plurality is Stablest Conjecture**, [9]). *Let $n \geq 2$, $k \geq 3$, $\rho \in [-\frac{1}{k-1}, 1]$, $\varepsilon > 0$. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^n$. Then there exists $\tau > 0$ such that, if $f\colon \{1,\ldots,k\}^n \to \Delta_k$ satisfies $\sum_{\sigma\in\{1,\ldots,k\}^n\colon \sigma_j\neq 0}(\widehat{f_i}(\sigma))^2 \leq \tau$ for all $i \in \{1,\ldots,k\}$, $j \in \{1,\ldots,n\}$, then*

(a) *If $\rho \in (0,1]$, and if $\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n} f(\sigma) = \frac{1}{k}\sum_{i=1}^{k} e_i$, then*

$$\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n} \langle f(\sigma), T_\rho f(\sigma)\rangle \leq \lim_{m\to\infty} \frac{1}{k^m}\sum_{\sigma\in\{1,\ldots,k\}^m} \langle \mathrm{PLUR}_{m,k}(\sigma), T_\rho(\mathrm{PLUR}_{m,k})(\sigma)\rangle + \varepsilon.$$

(b) *If $\rho \in (-1,0)$, then*

$$\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n} \langle f(\sigma), T_\rho f(\sigma)\rangle \geq \lim_{m\to\infty} \frac{1}{k^m}\sum_{\sigma\in\{1,\ldots,k\}^m} \langle \mathrm{PLUR}_{m,k}(\sigma), T_\rho(\mathrm{PLUR}_{m,k})(\sigma)\rangle - \varepsilon.$$

1.2. **A Synopsis of the Main Theorem.** We now describe the proof of Theorem 1.2. We first take the derivative $d/d\rho$ of the quantity J defined by (2). This procedure is common, and dates back at least to the the proof of the Log-Sobolev Inequality by Gross [7]. Taking this derivative allows us to relate J to the works [11, 12]. In Section 2, we modify the results of [11, 12] to prove the existence of a partition that maximizes $(d/d\rho)J$. Then, in Section 3, we further modify results of [11, 12] to show that, if $\rho > 0$ is small, then a partition maximizing $(d/d\rho)J$ is close to a partition maximizing $(d/d\rho)|_{\rho=0}J$. And by [11],

we know that the partition maximizing $(d/d\rho)|_{\rho=0}J$ is a regular simplicial conical partition, for dimension $n\geq 2$ and $k=3$ partition elements.

So, for small $\rho>0$, a partition maximizing $(d/d\rho)J$ is close to a regular simplicial conical partition. The structure of the operator T_ρ then permits the exploitation of a feedback loop. This feedback loop says: if our partition maximizes $(d/d\rho)J$ for small $\rho>0$, and if this maximal partition is close to a regular simplicial conical partition, then the maximal partition is even closer to a regular simplicial conical partition. This feedback loop is investigated in Section 4, especially in the crucial Lemma 4.4. A similar feedback loop was already apparent in [11][Lemma 3.3]. The full argument of Theorem 1.2 is then assembled in Section 5. By using this feedback loop, we show in Theorem 5.1 that a regular simplicial conical partition maximizes $(d/d\rho)J$ for small $\rho>0$, $k=3$, $n\geq 2$. Then, the Fundamental Theorem of Calculus allows us to relate $(d/d\rho)J$ to J, therefore completing the proof of the main theorem, Theorem 1.2.

In Section 5, we also show the surprising fact that our strategy fails for small negative correlation. That is, for small $\rho<0$, $(d/d\rho)J$ is not maximized by the regular simplicial conical partition. This result does not confirm or deny Conjecture 1 for $\rho<0$. However, one may interpret from this result that the case of Conjecture 1 for $\rho<0$ could be more difficult than the case $\rho>0$.

We should also emphasize the lack of symmetrization in the proof of Theorem 1.2. Symmetrization is one of a few general strategies that solves many optimization problems. In our context, symmetrization would appear as follows. Recall the definition of J from (2). Suppose we have a partition $\{A_i\}_{i=1}^k\subseteq\mathbb{R}^n$. Change this partition into a "more symmetric" partition $\{\widetilde{A_i}\}_{i=1}^k$ such that J or $(d/d\rho)J$ is larger for $\{\widetilde{A_i}\}_{i=1}^k$. In the proof of the main theorem, it is tempting to use this symmetrization paradigm. The works [3],[15] and [9] use Gaussian symmetrization in a crucial way. However, we find this approach to be less natural for Conjecture 1, so we do not explicitly use symmetrization. Nevertheless, symmetry does play a crucial role in our proof, especially in the estimates of Section 3. It should also be noted that the works [11, 12] do not explicitly use symmetrization, and this lack of symmetrization is one of their novel aspects.

1.3. **Preliminaries.** We follow the exposition of [14]. Let $n\geq 1$, $n\in\mathbb{Z}$. Let $\mathbb{N}=\{0,1,2,3,\ldots\}$. For $f\colon\mathbb{R}^n\to\mathbb{R}$ measurable, let $||f||_{L_2(\gamma_n)}:=(\int_{\mathbb{R}^n}|f|^2\,d\gamma_n)^{1/2}$. Let $L_2(\gamma_n):=\{f\colon\mathbb{R}^n\to\mathbb{R}\colon\,||f||_{L_2(\gamma_n)}<\infty\}$. Let ℓ_2^n denote the ℓ_2 metric on $\mathbb{R}^n$. For $x\in\mathbb{R}^n$ and $r>0$, define $B(x,r):=\{y\in\mathbb{R}^n\colon\,||x-y||_2<r\}$.

For $f\in L_2(\gamma_n)$, define T_ρ as in (1). T_ρ is a parametrization of the Ornstein-Uhlenbeck operator. T_ρ is not a semigroup, but it satisfies $T_{\rho_1}T_{\rho_2}=T_{\rho_1\rho_2}$, $\rho_1,\rho_2\in[-1,1]$, by (6) below. We use this definition since the usual Ornstein-Uhlenbeck operator is only defined for $\rho\in[0,1]$. Let $\lambda>0$, $x\in\mathbb{R}$. Recall that the Hermite polynomials of one variable are defined by the generating function

$$e^{\lambda x-\lambda^2/2}=:\sum_{\ell\in\mathbb{N}}\lambda^\ell h_\ell(x). \tag{4}$$

Alternatively, one defines the polynomials $H_\ell(x)$ such that $h_\ell(x)=2^{-\ell/2}(\ell!)^{-1}H_\ell(x/\sqrt{2})$. This convention is used in [1], where the orthogonality properties of the Hermite polynomials are derived.

Note that $\int_{\mathbb{R}} h_\ell^2 d\gamma_1 = 1/\ell!$, and $\{\sqrt{\ell!}\, h_\ell\}_{\ell\in\mathbb{N}}$ is an orthonormal basis of $L_2(\gamma_1)$. Set $f(x) := e^{\lambda x-\lambda^2/2}$. A routine computation shows that $T_\rho(f)(x) = e^{(\lambda\rho)x-(\lambda\rho)^2/2}$. Observe

$$\begin{aligned}
T_\rho(f)(x) &= \int e^{\lambda(x\rho+y\sqrt{1-\rho^2})-\lambda^2/2} d\gamma_1(y) = \int e^{(\lambda\rho)x+(\lambda\sqrt{1-\rho^2})y-\lambda^2/2-y^2/2} \frac{dy}{\sqrt{2\pi}} \\
&= e^{(\lambda\rho)x-\lambda^2/2} \int e^{-\frac{1}{2}(y-\lambda\sqrt{1-\rho^2})^2+\lambda^2(1-\rho^2)/2} \frac{dy}{\sqrt{2\pi}} = e^{(\lambda\rho)x-\lambda^2/2+\lambda^2(1-\rho^2)/2} \\
&= e^{(\lambda\rho)x-(\lambda\rho)^2/2}.
\end{aligned}$$

Therefore, by (4),

$$T_\rho f(x) = \sum_{\ell\in\mathbb{N}} \lambda^\ell \rho^\ell h_\ell(x). \tag{5}$$

So, by linearity, $T_\rho h_\ell(x) = \rho^\ell h_\ell(x)$.

We now extend the above observations to higher dimensions. Let $f \in L_2(\gamma_n)$, so that $f = \sum_{\ell\in\mathbb{N}^n} a_\ell h_\ell \sqrt{\ell!}$, $a_\ell \in \mathbb{R}$, where $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$ and $h_\ell(x) = \prod_{i=1}^n h_{\ell_i}(x_i)$. Write $|\ell| := \ell_1 + \cdots + \ell_n$ and $\ell! := (\ell_1!)\cdots(\ell_n!)$. Then T_ρ satisfies $T_\rho h_\ell = \rho^{|\ell|} h_\ell$, and for $x \in \mathbb{R}^n$,

$$T_\rho f(x) = \sum_{\ell\in\mathbb{N}^n} \rho^{|\ell|} \sqrt{\ell!}\, h_\ell(x) \left(\int \sqrt{\ell!}\, h_\ell f d\gamma_n \right) \tag{6}$$

Let $\Delta := \sum_{i=1}^n \partial^2/\partial x_i^2$. For $\rho \in (-1, 1)$, define $Lf := \frac{1}{\rho}(\langle x, \nabla f\rangle - \Delta f)$. For $\rho = 0$, L is undefined, but LT_ρ is defined. A well-known calculation shows that

$$\begin{aligned}
\frac{d}{d\rho} T_\rho f(x) = LT_\rho f(x) &= \frac{1}{\rho} \left(\langle x, \nabla T_\rho f(x)\rangle - \Delta T_\rho f(x) \right) &\quad (7) \\
&= \frac{1}{\sqrt{1-\rho^2}} \left[\left\langle x, \int_{\mathbb{R}^n} y f(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y) \right\rangle \right. \\
&\qquad \left. + \frac{\rho}{\sqrt{1-\rho^2}} \int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2) \right) f(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y). \right. &\quad (8)
\end{aligned}$$

We say that $A \subseteq \mathbb{R}^n$ is a **cone** if A is measurable and $\forall\, t > 0$, $tA = A$.

Definition 1.7. A **simplicial conical partition** $\{A_i\}_{i=1}^k$ is a partition of $\mathbb{R}^n$ together with a j-dimensional simplex $S \subseteq \mathbb{R}^j$ with $0 \le j \le n$ and a rotation σ of $\mathbb{R}^n$ such that $0 \in S$ and such that each face F_i of $\sigma(S \times \mathbb{R}^{n-j})$ generates a partition element, i.e. $A_i = \{tF_i : t \in [0, \infty)\}$, $i \in \{1, \ldots, j+1\}$. Let $k-1 \le n$ and let $\{z_i\}_{i=1}^k \subseteq \mathbb{R}^n$ be nonzero vectors that do not all lie in a $(k-1)$-dimensional hyperplane. Define a partition such that, for $i \in \{1, \ldots, k\}$, $A_i := \{x \in \mathbb{R}^n : \langle x, z_i\rangle = \max_{j=1,\ldots,k}\langle x, z_j\rangle\}$. Such a partition is called the simplicial conical partition induced by $\{z_i\}_{i=1}^k$.

If $\{A_i\}_{i=1}^k$ is a simplicial conical partition induced by the vectors $\{\int_{A_i} x d\gamma_n(x)\}_{i=1}^k$, then we say the partition is a **balanced conical partition**. If $\{z_i\}_{i=1}^k \subseteq \mathbb{R}^n$ are the vertices of a $(k-1)$-dimensional regular simplex in $\mathbb{R}^n$ centered at the origin, then the partition induced by $\{z_i\}_{i=1}^k$ is called a **regular simplicial conical partition**.

Let $f \in L_2(\gamma_n)$. By Plancherel and (6)

$$\int f T_\rho f d\gamma_n = \sum_{\ell \in \mathbb{N}^n} \rho^{|\ell|} \left| \int f \sqrt{\ell!} h_\ell d\gamma_n \right|^2. \tag{9}$$

Substituting (9) into (2) gives

$$\sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{A_i} T_\rho(1_{A_i}) d\gamma_n = \sum_{i=1}^{k} \left[\gamma_n(A_i)^2 + \rho \left|\left| \int_{A_i} x d\gamma_n(x) \right|\right|_{\ell_2^n}^2 + \sum_{\substack{\ell \in \mathbb{N}^n \\ |\ell| \geq 2}} \rho^{|\ell|} \left| \int 1_{A_i} \sqrt{\ell!} h_\ell d\gamma_n \right|^2 \right]. \tag{10}$$

Taking the derivative $d/d\rho$ of (10) at $\rho = 0$, we get a quantity studied in [11, 12].

$$\frac{d}{d\rho}\bigg|_{\rho=0} \sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{A_i} T_\rho(1_{A_i}) d\gamma_n = \sum_{i=1}^{k} \left|\left| \int_{A_i} x d\gamma_n(x) \right|\right|_{\ell_2^n}^2. \tag{11}$$

2. The First Variation

Definition 2.1. Let $H := \oplus_{i=1}^{k} L_2(\gamma_n)$, and define

$$\Delta_k(\gamma_n) := \{(f_1, \ldots, f_k) \in H \colon \forall\, 1 \leq i \leq k, 0 \leq f_i \leq 1, \sum_{i=1}^{k} f_i = 1\}. \tag{12}$$

Definition 2.2. Let $\varepsilon \geq 0$. Define

$$\Delta_k^\varepsilon(\gamma_n) := \{(f_1, \ldots, f_k) \in H \colon \forall\, 1 \leq i \leq k, 0 \leq f_i \leq 1, \sum_{i=1}^{k} f_i = 1, \frac{1}{k} - \varepsilon \leq \int f_i d\gamma_n \leq \frac{1}{k} + \varepsilon\}.$$

Definition 2.3. Define a metric d_2 on partitions $\{A_i\}_{i=1}^k, \{C_i\}_{i=1}^k$ of $\mathbb{R}^n$ by the formula

$$d_2(\{A_i\}_{i=1}^k, \{C_i\}_{i=1}^k) := \inf_{\substack{\rho \in SO(n) \\ \pi \text{ a permutation}}} \left(\sum_{i=1}^{k} \left|\left| 1_{A_i} - 1_{(\rho C_{\pi(i)})} \right|\right|_{L_2(\gamma_n)}^2 \right)^{1/2}.$$

Definition 2.4. Let $A \subseteq \mathbb{R}^n$, let $\mathcal{L}$ denote Lebesgue measure on $\mathbb{R}^n$, and define the distance $d(x, y) := ||x - y||_{\ell_2^n}$, $x, y \in \mathbb{R}^n$. Denote δ_A as the measure on $\mathbb{R}^n$ such that $\delta_A(B) := \liminf_{\delta \to 0} \frac{1}{2\delta} \mathcal{L}\{y \in \mathbb{R}^n \colon d(x, y) < \delta, x \in A \cap B\}$, $B \subseteq \mathbb{R}^n$. Also, we denote $\gamma_n(\delta_A) := \liminf_{\delta \to 0} \frac{1}{2\delta} \gamma_n\{y \in \mathbb{R}^n \colon d(y, A) < \delta\}$.

We now recall some results of [11], which will be used later, together with (11).

Lemma 2.5. [11, Lemma 3.2, Corollary 3.4] *Let $k = 3$, $n \geq 2$ and let $\{B_i\}_{i=1}^k$ be a regular simplicial conical partition of $\mathbb{R}^n$. Then $(1_{B_1}, \ldots, 1_{B_k})$ uniquely achieves the following supremum, up to orthogonal transformation.*

$$\sup_{(f_1, \ldots, f_k) \in \Delta_k(\gamma_n)} \sum_{i=1}^{k} \left|\left| \int_{\mathbb{R}^n} x f_i(x) d\gamma_n(x) \right|\right|_{\ell_2^n}^2.$$

Lemma 2.6. [11, Lemma 3.2, Corollary 3.4] *Let $k = 3$, $n \geq 2$ and let $\{B_i\}_{i=1}^k \subseteq \mathbb{R}^n$ be a regular simplicial conical partition. Let $\{C_i\}_{i=1}^2 \subseteq \mathbb{R}^n$ be a regular simplicial conical partition, and let $\{\widetilde{A_i}\}_{i=1}^k \subseteq \mathbb{R}^n$ be a simplicial conical partition. Let $\widetilde{z_i} := \int_{\widetilde{A_i}} x d\gamma_n(x)$, and let $v_{ij} \in S^{n-1} \cap \in \widetilde{A_i} \cap \widetilde{A_j} \cap \mathrm{span}\{\widetilde{z_i}, \widetilde{z_j}\}$. If $|\langle \widetilde{z_i} - \widetilde{z_j}, v_{ij}\rangle| \leq \varepsilon < 10^{-16}$ $\forall$ $i, j \in \{1, \ldots, k\}$, $i \neq j$, and if $d_2(\{\widetilde{A_i}\}_{i=1}^k, \{C_1, C_2, \emptyset\}) > 1/100$, then $d_2(\{\widetilde{A_i}\}_{i=1}^k, \{B_i\}_{i=1}^k) \leq \sqrt{6\varepsilon}$.*

Proof. For $i, j \in \{1, \ldots, k\}$, let $0 \leq \alpha_i \leq \pi$ such that $\widetilde{A_i}$ is a cone with angle α_i. Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ be a reflection that fixes $\widetilde{A_i} \cap \widetilde{A_j}$. Without loss of generality, $\sigma(\widetilde{A_j}) \subseteq \widetilde{A_i}$. Then $\widetilde{z_i} - \widetilde{z_j} = \int_{\widetilde{A_i} \setminus \sigma(\widetilde{A_j})} x d\gamma_n(x)$ and $||\widetilde{z_i} - \widetilde{z_j}||_2 = \sin((\alpha_i - \alpha_j)/2)/\sqrt{2\pi}$. Let $0 \leq \theta \leq \pi$ such that $||\widetilde{z_i} - \widetilde{z_j}||_2 \cos(\theta) = \langle \widetilde{z_i} - \widetilde{z_j}, v_{ij}\rangle$. Then either $||\widetilde{z_i} - \widetilde{z_j}||_2 \leq \sqrt{\varepsilon/18\pi}$, or $|\cos\theta| \leq \sqrt{18\pi\varepsilon}$. In the first case, $\alpha_i - \alpha_j \leq \sqrt{\varepsilon}$. So, to complete the proof, it suffices to show that the second case does not occur. We find a contradiction by assuming that the second case occurs.

If $|\cos\theta| \leq \sqrt{18\pi\varepsilon}$, then since $\theta = (\alpha_i - \alpha_j)/2$, we must have $|\alpha_i - \alpha_j - \pi| < 18\sqrt{\varepsilon}$, so $\pi - 18\sqrt{\varepsilon} < \alpha_i - \alpha_j < \pi + 18\sqrt{\varepsilon}$, i.e. $\pi - 18\sqrt{\varepsilon} < \alpha_i \leq \pi$ and $\alpha_j \leq 18\sqrt{\varepsilon}$. Then for $r \neq i, j$, $r \in \{1, \ldots, k\}$, we have $\alpha_r = 2\pi - \alpha_i - \alpha_j > 2\pi - \pi - 18\sqrt{\varepsilon} > \pi - 18\sqrt{\varepsilon}$. Since $\alpha_i, \alpha_j > \pi - 18\sqrt{\varepsilon}$, we conclude that $d_2(\{\widetilde{A_i}\}_{i=1}^k, \{C_1, C_2, \emptyset\}) < 18\varepsilon^{1/4} < 1/100$, a contradiction. $\square$

The following is a variant of an argument from [11, Lemma 3.3] and [12, Lemma 2.1].

Lemma 2.7. *Let $\rho \in (0, 1]$. Then there exists a partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ that maximizes (2). Moreover, the following containment holds, less sets of γ_n measure zero:*

$$A_i \supseteq \{x \in \mathbb{R}^n \colon LT_\rho 1_{A_i}(x) > LT_\rho 1_{A_j}(x), \forall\, j \neq i, j \in \{1, \ldots, k\}\}. \tag{13}$$

There also exists a partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ that maximizes (2), subject to the constraint $\gamma_n(A_i) = 1/k$.

Proof. We first consider the supremum of (2) without volume constraints. We show that (2) is maximized over $\Delta_k(\gamma_n)$, which contains the set of partitions of $\mathbb{R}^n$. Note that $\Delta_k(\gamma_n) \subseteq H$ is norm closed, convex, and norm bounded. Therefore, $\Delta_k(\gamma_n)$ is weakly closed. Also, $\Delta_k(\gamma_n)$ is weakly compact by Banach-Alaoglu. Using (7), define a map $\psi_\rho\colon \Delta_k(\gamma_n) \to \mathbb{R}$ by

$$\psi_\rho(g_1, \ldots, g_k) := \frac{d}{d\rho} \sum_{i=1}^k \int g_i T_\rho g_i d\gamma_n = \sum_{i=1}^k \int g_i LT_\rho g_i d\gamma_n. \tag{14}$$

By (10), ψ_ρ is an exponentially decaying sum of uniformly bounded weakly continuous functions. Therefore, ψ_ρ is weakly continuous on the weakly compact set $\Delta_k(\gamma_n)$. So there exists $(f_1, \ldots, f_k) \in \Delta_k(\gamma_n)$ that maximizes ψ_ρ.

Since $\rho \in (0, 1]$, (9) and (7) imply: $\forall$ $f \in L_2(\gamma_n)$, $\int f LT_\rho f d\gamma_n \geq 0$. We now apply this fact to see that ψ_ρ is convex. Let $\lambda \in [0, 1]$, $(g_1, \ldots, g_k), (h_1, \ldots, h_k) \in \Delta_k(\gamma_n)$. Then

$$\begin{aligned}
&\lambda\psi_\rho(g_1, \ldots, g_k) + (1-\lambda)\psi_\rho(h_1, \ldots, h_k) - \psi_\rho(\lambda g_1 + (1-\lambda)h_1, \ldots, \lambda g_k + (1-\lambda)h_k) \\
&\quad = \sum_{i=1}^k \left[\lambda \int g_i LT_\rho g_i + (1-\lambda) \int h_i LT_\rho h_i - \int (\lambda g_i - (1-\lambda)h_i) LT_\rho (\lambda g_i - (1-\lambda)h_i)\right] \\
&\quad = \lambda(1-\lambda) \int (g_i - h_i) LT_\rho (g_i - h_i) \geq 0.
\end{aligned}$$

Since ψ_ρ is convex on $\Delta_k(\gamma_n)$, ψ_ρ achieves its maximum at an extreme point of $\Delta_k(\gamma_n)$. Therefore, there exists a partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ such that $(1_{A_1},\ldots,1_{A_k})\in\Delta_k(\gamma_n)$ maximizes ψ_ρ on $\Delta_k(\gamma_n)$ [12, Lemma 2.1].

We now prove (13) by contradiction. By the Lebesgue density theorem [17][1.2.1, Proposition 1], we may assume that, for all $i\in\{1,\ldots,k\}$, if $y\in A_i$, then we have $\lim_{r\to 0}\gamma_n(A_i\cap B(y,r))/\gamma_n(B(y,r))=1$. Suppose there exist $j,m\in\{1,\ldots,k\}$ and there exists $y\in\mathbb{R}^n$, $r>0$ such that $y\in A_j$, $\gamma_n(B(y,r)\cap A_j)>0$, and $LT_\rho 1_{A_j}(y)<LT_\rho 1_{A_m}(y)$. By (1),

$$T_\rho f(x)=\int_{\mathbb{R}^n} f(y)e^{-||y-x\rho||_2^2/[2(1-\rho^2)]}\frac{dy}{(2\pi(1-\rho^2))^{n/2}}.$$

So, $LT_\rho 1_{A_j}$ is a smooth function of x, and for any ball B, $||1_B\nabla^\ell LT_\rho(1_{A_j}-1_{A_m})||_\infty\leq C(B,\ell)||1_{A_j}-1_{A_m}||_\infty$, $\forall\,\ell\in\mathbb{N}$. Here ∇^ℓ is the array of all partial derivatives of order ℓ.

So, there exists a ball $B(y,r)$, $r>0$ such that $\gamma_n(B(y,r)\cap A_j)>0$ and such that

$$\sup_{x\in B(y,r)} LT_\rho 1_{A_j}(x)<\inf_{x\in B(y,r)} LT_\rho 1_{A_m}(x).$$

Let $\phi(x):=1_{B(y,r)\cap A_j}(x)$. For $\lambda\in[0,1]$, note that

$$\big(1_{A_1},\ldots,1_{A_j}-\lambda\phi,\ldots,1_A+\lambda\phi,\ldots,1_{A_k}\big)\in\Delta_k(\gamma_n). \tag{15}$$

However,

$$\frac{d}{d\lambda}\bigg|_{\lambda=0}\psi_\rho(1_{A_1},\ldots,1_{A_j}-\lambda\phi,\ldots,1_{A_m}+\lambda\phi,\ldots,1_{A_k})=2\int\phi(LT_\rho 1_{A_m}-LT_\rho 1_{A_j})d\gamma_n>0. \tag{16}$$

But (16) contradicts the maximality of $(1_{A_1},\ldots,1_{A_k})$ on $\Delta_k(\gamma_n)$, so (13) holds.

Finally, in the case that we maximize (2) subject to the constraint $\gamma_n(A_i)=1/k$, we replace $\Delta_k(\gamma_n)$ with $\Delta_k^0(\gamma_n)$. The existence argument then proceeds as shown above. □

3. Perturbative Estimates

The following estimates allow us to relate ψ_ρ to ψ_0 for small $\rho>0$.

Lemma 3.1. *Let $A\subseteq\mathbb{R}^n$ be a cone. Then*

$$\int_{\mathbb{R}^n}\left(\sum_{i=1}^n(1-y_i^2)\right)1_A(y)d\gamma_n(y)=0.$$

Proof. The assertion follows by standard equalities for the moments of a Gaussian random variable. Let $\alpha>0$. Define $f(\alpha)$ by the formula

$$f(\alpha):=\int_{\mathbb{R}^n}1_A(y)e^{-\alpha(y_1^2+\cdots+y_n^2)/2}\frac{dy}{(2\pi)^{n/2}}.$$

By changing variables, $f(\alpha)=\alpha^{-n/2}\int_{\mathbb{R}^n}1_A(y)d\gamma_n(y)$. So,

$$-\frac{1}{2}\int_{\mathbb{R}^n}\left(\sum_{i=1}^n y_i^2\right)1_A(y)d\gamma_n(y)=\frac{df(\alpha)}{d\alpha}\bigg|_{\alpha=1}=-\frac{n}{2}\int_{\mathbb{R}^n}1_A(y)d\gamma_n(y).$$

□

Lemma 3.2. *Fix $k=3$, $n\geq 2$, $\rho\in(0,1)$. Let $\{C_i\}_{i=1}^k\subseteq\mathbb{R}^2$ be a simplicial conical partition. Let $\{B_i\}_{i=1}^k:=\{C_i\times\mathbb{R}^{n-2}\}_{i=1}^k$. Fix $i,j\in\{1,\ldots,k\}$. Let $\sigma\colon\mathbb{R}^n\to\mathbb{R}^n$ denote reflection across $B_i\cap B_j$. Assume that $B_i=\sigma B_j$, and that $B_i\subseteq\{x\in\mathbb{R}^n\colon x_1\geq 0\}$. Let $e_1=(1,0,\ldots,0)$, $e_2=(0,1,0,\ldots,0)$, $e_1,e_2\in\mathbb{R}^n$. For $p\in\{1,\ldots,k\}$, let $z_p:=\int_{B_p}xd\gamma_n(x)$. Note that $\mathrm{span}\{z_i,z_j\}=\mathrm{span}\{e_1,e_2\}$. Let $n_j\in\mathbb{R}^n$ be the interior unit normal of B_j so that n_j is normal to the face $(\partial B_j)\setminus(\partial B_i)$, and let $n_i\in\mathbb{R}^n$ be the interior unit normal of B_i so that n_i is normal to the face $(\partial B_i)\setminus(\partial B_j)$.*

If $x\in B_i\cap\{x\in\mathbb{R}^n\colon\langle x,n_j\rangle\leq 0\}$, then

$$\frac{1}{\rho}\left\langle x,\nabla T_\rho(1_{B_i}-1_{B_j})(x)\right\rangle\geq 2x_1\gamma_n\left(\delta_{\frac{(B_i\cap B_j)-x\rho}{\sqrt{1-\rho^2}}}\right)+\langle x,n_i\rangle\gamma_n\left(\delta_{\frac{((\partial B_i)\setminus B_j)-x\rho}{\sqrt{1-\rho^2}}}\right). \tag{17}$$

If $x\in B_i\cap\{x\in\mathbb{R}^n\colon\langle x,n_j\rangle\geq 0\}$, then

$$\frac{1}{\rho}\left\langle x,\nabla T_\rho(1_{B_i}-1_{B_j})(x)\right\rangle\geq 2x_1\gamma_n\left(\delta_{\frac{(B_i\cap B_j)-x\rho}{\sqrt{1-\rho^2}}}\right). \tag{18}$$

Also, for $x\in B_i$,

$$\left|\int_{\mathbb{R}^n}\left(\sum_{i=1}^n(1-y_i^2)\right)(1_{B_i}-1_{B_j})(x\rho+y\sqrt{1-\rho^2})d\gamma_n(y)\right|\leq\rho(\sqrt{6}+(n-1)\sqrt{2})x_1. \tag{19}$$

And for $x\in B_i$ with $x_1>\sqrt{n}\sqrt{1-\rho^2}/\rho$,

$$\int_{\mathbb{R}^n}\left(\sum_{i=1}^n(1-y_i^2)\right)(1_{B_i}-1_{B_j})(x\rho+y\sqrt{1-\rho^2})d\gamma_n(y)\geq 0. \tag{20}$$

Proof. Let $x\neq 0$. For $x\in(\partial B_i)\cap B_j$, $\nabla 1_{B_i}(x)=e_1$, and for $x\in(\partial B_i)\setminus B_j$, $\nabla 1_{B_i}(x)=n_i$. Similarly, for $x\in(\partial B_j)\cap B_i$, $-\nabla 1_{B_j}(x)=e_1$, and for $x\in(\partial B_j)\setminus B_i$, $-\nabla 1_{B_j}(x)=-n_j$. Then from (1),

$$\begin{aligned}
&\frac{1}{\rho}\nabla T_\rho(1_{B_i}-1_{B_j})(x)=T_\rho(\nabla(1_{B_i}-1_{B_j}))(x)\\
&=T_\rho[2(e_1)\delta_{B_i\cap B_j}+n_i\delta_{(\partial B_i)\setminus B_j}+(-n_j)\delta_{(\partial B_j)\setminus B_i}](x)\\
&=2e_1\gamma_n\left(\delta_{\frac{(B_i\cap B_j)-x\rho}{\sqrt{1-\rho^2}}}\right)+n_i\gamma_n\left(\delta_{\frac{((\partial B_i)\setminus B_j)-x\rho}{\sqrt{1-\rho^2}}}\right)+(-n_j)\gamma_n\left(\delta_{\frac{((\partial B_j)\setminus B_i)-x\rho}{\sqrt{1-\rho^2}}}\right).
\end{aligned}\tag{21}$$

Here we used

$$\int 1_A(x\rho+y\sqrt{1-\rho^2})d\gamma_n(x)=\int 1_{A-x\rho}(y\sqrt{1-\rho^2})d\gamma_n(y)=\int 1_{(A-x\rho)/\sqrt{1-\rho^2}}(y)d\gamma_n(y).$$

Let x with $x\in B_i$ and $\langle x,(-n_j)\rangle\geq 0$. Then (21) immediately proves (17).

Now, let $x\in B_i\cap\{x\in\mathbb{R}^n\colon\langle x,n_j\rangle\geq 0\}$. By reflecting across $B_i\cap B_j$,

$$\gamma_n\left(\delta_{\frac{((\partial B_i)\setminus B_j)-x\rho}{\sqrt{1-\rho^2}}}\right)\geq\gamma_n\left(\delta_{\frac{((\partial B_j)\setminus B_i)-x\rho}{\sqrt{1-\rho^2}}}\right). \tag{22}$$

Define

$$w:=n_i\gamma_n\left(\delta_{\frac{((\partial B_i)\setminus B_j)-x\rho}{\sqrt{1-\rho^2}}}\right)+(-n_j)\gamma_n\left(\delta_{\frac{((\partial B_j)\setminus B_i)-x\rho}{\sqrt{1-\rho^2}}}\right).$$

By (22), $w \in \text{hull}\{e_1, n_i\}$. In particular, $\langle x, w\rangle \geq 0$, since $x \in B_i$. Combining $\langle x, w\rangle \geq 0$ with (21) proves (18).

We now prove (19). By reflecting across $B_i \cap B_j$,

$$x \in B_i \cap B_j \quad \Longrightarrow \quad \int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y) = 0.$$

So, a derivative bound gives (19). Specifically, we apply the Fundamental Theorem of Calculus to the following identity, along with $||(1-y_2^2)y_1||_{L_2(\gamma_n)} = \sqrt{2}$ and $||y_1^3 + 3y_1||_{L_2(\gamma_n)} = \sqrt{6}$.

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\\
&\quad= \frac{\rho}{\sqrt{1-\rho^2}}\int_{\mathbb{R}^n} \left(-3y_1 - y_1^3 + \sum_{i\neq 1}(1-y_i^2)y_1\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y).
\end{aligned}$$

Let x with $x_1 > \sqrt{n}\sqrt{1-\rho^2}/\rho$. To prove (20), consider the following cone

$$A = \{0\} \cup \left\{y \in \mathbb{R}^n \colon y \neq 0 \,\wedge\, \sqrt{n}\frac{y}{||y||_2} \in \frac{B_i - x\rho}{\sqrt{1-\rho^2}}\right\}.$$

By Lemma 3.2, $\int_{\mathbb{R}^n} 1_A(y)\sum_{i=1}^n(1-y_i^2)d\gamma_n(y) = 0$. If $d(x, \partial B_i) \geq \sqrt{n}\sqrt{1-\rho^2}/\rho$, then $A = \mathbb{R}^n$ and $1_A(y)\sum_{i=1}^n(1-y_i^2)1_{B_i^c}(x\rho + y\sqrt{1-\rho^2}) \leq 0$, so

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\\
&\qquad\geq \int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)1_{B_i}(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y) \geq \int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)d\gamma_n(y) = 0.
\end{aligned}$$

So, it remains to consider the case $d(x, \partial B_i) < \sqrt{n}\sqrt{1-\rho^2}/\rho$. In this case $A \neq \mathbb{R}^n$.

Since $x_1 > \sqrt{n}\sqrt{1-\rho^2}/\rho$, we have $\sum_{i=1}^n(1-y_i^2)1_{B_j}(x\rho + y\sqrt{1-\rho^2}) \leq 0$. Also, we have $\sum_{i=1}^n(1-y_i^2)1_{A^c}(y)1_{B_i}(x\rho + y\sqrt{1-\rho^2}) \geq 0$, $\sum_{i=1}^n(1-y_i^2)1_A(y)1_{B_i^c}(x\rho + y\sqrt{1-\rho^2}) \leq 0$, so

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\\
&\qquad\geq \int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)1_{B_i}(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\\
&\qquad\geq \int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)1_A(y)1_{B_i}(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\\
&\qquad\geq \int_{\mathbb{R}^n} \left(\sum_{i=1}^n (1-y_i^2)\right)1_A(y)d\gamma_n(y) = 0.
\end{aligned}$$

□

Lemma 3.3. *Let $\varepsilon > 0$, $k = 3$, $n \geq 2$. Let $\{A_i\}_{i=1}^k$ be a partition of $\mathbb{R}^n$, and let $\{B_i\}_{i=1}^k$ be a regular simplicial conical partition of $\mathbb{R}^n$. Define ψ_0 by (14). Assume that*

$$\psi_0(1_{A_1}, \ldots, 1_{A_k}) > \sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)} \psi_0(f_1, \ldots, f_k) - \varepsilon. \tag{23}$$

Then there exists $\varepsilon_1 > 1/100$ such that, for $0 < \varepsilon < \varepsilon_1$,

$$d_2(\{A_i\}_{i=1}^k, \{B_i\}_{i=1}^k) \leq 6\varepsilon^{1/8}. \tag{24}$$

Proof. Assume that (23) holds. For $i \in \{1, \ldots, k\}$, let $z_i := \int_{A_i} x d\gamma_n(x)$, $w_i := \int_{B_i} x d\gamma_n(x)$. Let $\varepsilon_1 = 10^{-2}$. We may assume that, for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $\langle z_i, z_j\rangle < 0$. To see this, we argue by contradiction. Suppose there exist $i, j \in \{1, \ldots, k\}$, $i \neq j$ with $\langle z_i, z_j\rangle \geq 0$. For $p \in \{1, \ldots, k\}$, $p \neq i, j$, let $A_p'' := A_p$, let $A_i'' := A_i \cup A_j$, and let $A_j'' := \emptyset$. For $p \in \{1, \ldots, k\}$, let $z_p'' := \int_{A_p''} x d\gamma_n(x)$. Then

$$\sum_{p=1}^k ||z_p''||_{\ell_2^n}^2 - \sum_{p=1}^k ||z_p||_{\ell_2^n}^2 = ||z_i + z_j||_{\ell_2^n}^2 - ||z_i||_{\ell_2^n}^2 - ||z_j||_{\ell_2^n}^2 \geq 0.$$

That is,

$$\psi_0(1_{A_1}, \ldots, 1_{A_k}) \leq \psi_0(1_{A_1''}, \ldots, 1_{A_k''}). \tag{25}$$

Since $\{A_p''\}_{p=1}^k$ is a partition of $\mathbb{R}^n$ with at most two nonempty elements, [11][Corollary 3.4] implies that

$$\left(\sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)} \psi_0(f_1, \ldots, f_k)\right) - \psi_0(1_{A_1''}, \ldots, 1_{A_k''}) \geq \frac{1}{8\pi} > 10^{-2} = \varepsilon_1. \tag{26}$$

Combining (25) and (23) contradicts (23). Therefore, $\langle z_i, z_j\rangle < 0$ for all $i, j \in \{1, \ldots, k\}$.

We now claim that, for each pair $i, j \in \{1, \ldots, k\}$ with $i \neq j$, we have

$$\max_{p\in\{i,j\}} ||z_p||_{\ell_2^n}^2 \geq 1/16. \tag{27}$$

We again argue by contradiction. Suppose there exist $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and $\max_{p\in\{i,j\}} ||z_p||_{\ell_2^n}^2 < 1/16$. Let $p \in \{1, \ldots, k\}$, $p \neq i, j$. Then $||z_p||_{\ell_2^n} \leq 1/(2\pi)$ with equality if and only if 1_{A_p} is a half space whose boundary contains the origin of $\mathbb{R}^n$. This follows from [11][Lemma 3.3, Corollary 3.4]. Therefore,

$$\psi_0(1_{A_1}, \ldots, 1_{A_k}) \leq 1/8 + 1/(2\pi) \leq 1/\pi.$$

As before, this inequality contradicts (23), since $\sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)} \psi_0(f_1, \ldots, f_k) = 9/(8\pi)$, using [11][Corollary 3.4] and $k = 3$. We conclude that (27) holds.

Define δ such that

$$\delta := \max_{i,j\in\{1,\ldots,k\}, i\neq j} \gamma_n(\{x \in \mathbb{R}^n : \langle z_i - z_j, x\rangle \leq 0\} \cap A_i). \tag{28}$$

Fix $i, j \in \{1, \ldots, k\}$ such that $\delta = \gamma_n(\{x \in \mathbb{R}^n : \langle z_i - z_j, x\rangle \leq 0\} \cap A_i)$. We want to find a bound on δ. Let $0 < h$ such that $\int_0^h d\gamma_1 = \delta$. Now, define $\{A_r'\}_{r=1}^k$ such

that $A'_p = A_p$ for $p \neq i,j$, $A'_i = A_i \setminus (A_i \cap \{x \in \mathbb{R}^n \colon \langle z_i - z_j, x\rangle \leq 0\})$ and $A'_j = A_j \cup (A_i \cap \{x \in \mathbb{R}^n \colon \langle z_i - z_j, x\rangle \leq 0\})$. Let $z'_p := \int_{A'_p} x d\gamma_n(x)$, $p = 1, \ldots, k$. Then

$$\begin{aligned}
&\sum_{p=1}^{k} ||z'_p||_{\ell_2^n}^2 - \sum_{p=1}^{k} ||z_p||_{\ell_2^n}^2 \\
&\qquad = 2\left\langle \int_{\{y\colon \langle z_i - z_j, y\rangle \leq 0\}\cap A_i} y d\gamma_n(y), z_j - z_i \right\rangle + \left|\left| \int_{\{y\colon \langle z_i - z_j, y\rangle \leq 0\}\cap A_i} y d\gamma_n(y) \right|\right|_{\ell_2^n}^2 \qquad (29) \\
&\qquad \geq 2\left\langle \int_{\{y\colon -h \leq \langle z_i - z_j, y\rangle \leq 0\}} y d\gamma_n(y), z_j - z_i \right\rangle = 2\,||z_i - z_j||_{\ell_2^n} \int_0^h y d\gamma_1(y) \overset{(27)}{>} \delta^2/3.
\end{aligned}$$

Here we used rearrangement and also the inequality $||z_i - z_j||_{\ell_2^n} \geq (\max_{p\in\{i,j\}} ||z_p||_{\ell_2^n}^2)^{1/2}$.

By (23) and (29), $\delta^2 < \varepsilon$, i.e.

$$\delta < \sqrt{3\varepsilon}. \qquad (30)$$

Now, for $p \in \{1, \ldots, k\}$, let $\widetilde{A_p} := \{x \in \mathbb{R}^n \colon \langle x, z_p\rangle = \max_{j=1,\ldots,n}\langle x, z_j\rangle\}$ and let $\widetilde{z_p} := \int_{\widetilde{A_p}} x d\gamma_n(x)$. By (30), and (28),

$$d_2(\{A_i\}_{i=1}^k, \{\widetilde{A_i}\}_{i=1}^k) \leq 3\sqrt{2}\,\varepsilon^{1/4}. \qquad (31)$$

For $p \in \{1, \ldots, k\}$, let $y_p := \widetilde{z_p} - z_p \in \mathbb{R}^n$, so that $||y_p||_2 \leq 3\sqrt{2}\,\varepsilon^{1/4}$ by (31) and Hilbert space duality. Let $x \in \mathbb{R}^n$. Then for $i,j \in \{1, \ldots, k\}$, $i \neq j$,

$$\langle \widetilde{z_i} - \widetilde{z_j}, x\rangle = \langle z_i - z_j, x\rangle + \langle y_i - y_j, x\rangle. \qquad (32)$$

For $i,j \in \{1, \ldots, k\}$, $i \neq j$, let $v_{ij} = S^{n-1} \cap \widetilde{A_i} \cap \widetilde{A_j} \cap \operatorname{span}\{\widetilde{z_i}\}_{i=1}^k$. By definition of v_{ij} and $\{\widetilde{A_i}\}_{i=1}^k$, $\langle z_i - z_j, v_{ij}\rangle = 0$. So, by (32), $|\langle \widetilde{z_i} - \widetilde{z_j}, v_{ij}\rangle| \leq 3\sqrt{2}\,\varepsilon^{1/4}$, implying that $d_2(\{\widetilde{A_i}\}_{i=1}^k, \{B_i\}_{i=1}^k) \leq 3 \cdot 2^{3/4}\varepsilon^{1/8}$, by Lemma 2.6. This inequality together with (31) and the triangle inequality for d_2 prove (24). $\square$

4. Iterative Estimates

The following estimates control the errors that appear in the proof of Theorem 1.2.

Lemma 4.1. *For $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,*

$$h_\ell(x)\sqrt{\ell!} \leq |\ell|^n\, 3^{|\ell|} \prod_{i=1}^{k} \max\{1, |x_i|^{\ell_i}\}.$$

Proof. Let $\ell \in \mathbb{N}$. From (4),

$$\sum_{\ell=0}^{\infty} \lambda^\ell h_\ell(x) = e^{\lambda x - \lambda^2/2} = \sum_{p=0}^{\infty} \frac{x^p}{p!}\lambda^p \sum_{q=0}^{\infty} \frac{(-1)^q(\lambda)^{2q}(1/2)^q}{q!} = \sum_{\ell=0}^{\infty} \lambda^\ell \sum_{m=0}^{\lfloor \ell/2\rfloor} \frac{x^{\ell-2m}(-1)^m 2^{-m}}{m!(\ell-2m)!}.$$

Here we let $p + 2q = \ell$, $m = q$. In particular,

$$h_\ell(x) = \sum_{m=0}^{\lfloor \ell/2\rfloor} \frac{x^{\ell-2m}(-1)^m 2^{-m}}{m!(\ell-2m)!}. \qquad (33)$$

Using Stirling's formula, $\sqrt{2\pi}\ell^{\ell+1/2}e^{-\ell} \leq \ell! \leq e\,\ell^{\ell+1/2}e^{-\ell}$. Let ℓ, m with $m \in \{0, \ldots, \ell/2\}$, $\ell \geq 1$. Note that $\lim_{x\to 0^+} x^x = 1$ and $\min_{x\in[0,1]} x^x > 2/3$. Also, $m + \ell - 2m = \ell - m \geq \ell/2$. For $m \neq 0$, write $m = \ell j$, $j \in [1/\ell, 1/2]$. Moreover, $\max\{m, \ell - 2m\} \geq \ell/3$. Then

$$
\begin{aligned}
&\frac{\sqrt{\ell!}}{m!(\ell-2m)!} \\
&\leq \frac{\sqrt{e}\,\ell^{(1/2)\ell+1/4}e^{-\ell/2}}{2\pi m^{m+1/2}e^{-m}(\ell-2m)^{\ell-2m+1/2}e^{-(\ell-2m)}} = \frac{\sqrt{e}}{2\pi}\frac{\ell^{(1/2)\ell+1/4}}{m^{m+1/2}(\ell-2m)^{\ell-2m+1/2}}e^{\ell/2-m} \\
&= \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{\ell^{\ell/2}}{m^m(\ell-2m)^{\ell-2m}}e^{\ell/2-m} = \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{\ell^{\ell/2}}{(\ell j)^{\ell j}(\ell(1-2j))^{\ell(1-2j)}}e^{\frac{\ell}{2}-m} \\
&= \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{\ell^{\ell/2}e^{\ell/2-m}}{\ell^{\ell(1-j)}j^{\ell j}(1-2j)^{\ell(1-2j)}} = \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{\ell^{\ell/2}e^{\ell/2-m}}{\ell^{\ell/2}\ell^{\frac{\ell}{2}(1-2j)}j^{\ell j}(1-2j)^{\ell(1-2j)}} \\
&= \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{1}{j^{\ell j}(1-2j)^{\ell(1-2j)}}\frac{e^{\frac{\ell}{2}(1-2j)}}{\ell^{\frac{\ell}{2}(1-2j)}} = \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{(e/\ell)^{\frac{\ell}{2}(1-2j)}}{j^{\ell j}(1-2j)^{\ell(1-2j)}} \\
&\leq \frac{e}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{1}{j^{\ell j}(1-2j)^{\ell(1-2j)}} \leq \frac{e}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{1}{(2/3)^{2\ell}} \\
&\leq \frac{e\sqrt{3}}{\ell^{1/4}2\pi}\frac{1}{(2/3)^{2\ell}} = \frac{e\sqrt{3}}{\ell^{1/4}2\pi}(9/4)^{\ell} \leq \ell^{-1/4}(9/4)^{\ell}.
\end{aligned}
$$

Here we used $(e/\ell)^{(\ell/2)(1-2j)} \leq \sqrt{e}$ for $\ell = 1, 2$.

Also, for $m = 0$ we have $\frac{\sqrt{\ell!}}{m!(\ell-2m)!} = 1$, and for $m = \ell/2$ we have

$$
\begin{aligned}
\frac{\sqrt{\ell!}}{m!(\ell-2m)!} &= \frac{\sqrt{\ell!}}{(\ell/2)!} \leq \frac{\sqrt{e}\ell^{\ell/2+1/4}e^{-\ell/2}}{\sqrt{2\pi}(\ell/2)^{\ell/2+1/2}e^{-\ell/2}} = \sqrt{e}\frac{\ell^{1/4}}{\sqrt{2\pi}\ell^{1/2}2^{-\ell/2}2^{-1/2}} \\
&= \sqrt{\frac{e}{\pi}}\ell^{-1/4}2^{\ell/2} \leq \ell^{-1/4}2^{\ell/2}.
\end{aligned}
$$

So, combining the above estimates with (33),

$$
\begin{aligned}
|h_\ell(x)\sqrt{\ell!}| &\leq \sum_{m=0}^{\lfloor \ell/2 \rfloor} \ell^{-1/4}(9/4)^{\ell}\,|x|^{\ell-2m} \leq \sum_{m=0}^{\lfloor \ell/2 \rfloor} \ell^{-1/4}(9/4)^{\ell}\max\{1, |x|^{\ell-2m}\} \\
&\leq \ell\,\ell^{-1/4}(9/4)^{\ell}\max\{1, |x|^{\ell}\} \leq \ell\,3^{\ell}\max\{1, |x|^{\ell}\}.
\end{aligned}
$$

Therefore, for $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$,

$$
h_\ell(x)\sqrt{\ell!} \leq \ell_1 \cdots \ell_n 3^{\ell_1+\cdots+\ell_n}\prod_{i=1}^{k}\max\{1, |x_i|^{\ell_i}\} \leq |\ell|^n\,3^{|\ell|}\prod_{i=1}^{k}\max\{1, |x_i|^{\ell_i}\}.
$$

□

The following Lemma uses standard tail bounds for a Gaussian random variable. We therefore omit the proof.

Lemma 4.2. *Let $\eta > 0, t > 0$, and let $n \geq 2$. Then*

$$\Big|\int_{[-\eta,\eta]\times[t,\infty]\times\mathbb{R}^{n-2}} \sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}} \prod_{i=1}^{n} |y_i|^{\ell_i}\, d\gamma_n(y)\Big| \leq 3000n^3\eta(t^2+2)e^{-t^2/2},$$

$$\Big|\int_{B(0,t)^c} \sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}} \prod_{i=1}^{n} |y_i|^{\ell_i}\, d\gamma_n(y)\Big| \leq 4n^3 2^{-n/2}(\Gamma(n/2))^{-1}(n+2)!(t^{n+1}+1)e^{-t^2/2}$$
$$\leq 100(n+2)!(t^{n+1}+1)e^{-t^2/2}.$$

Lemma 4.3. *Let $\rho \in (-1,1)$, $n \geq 2$. Suppose $f \in L_2(\gamma_n)$ with $\int_{\mathbb{R}^n} y_2 f(y)d\gamma_n(y) = 0$. Let $x_1 \geq 0$ and $x_2 \geq 0$. Then*

$$\begin{aligned}\left|\frac{d}{d\rho}T_\rho f(x_1,x_2,0,\ldots,0)\right| &\leq \big[|x_1| + 2\rho(|x_2|^2 + (n+1)\rho\,|x_2| + |x_1x_2| + 2n)\big] \\ &\cdot \sup_{\substack{t_1\in[0,x_1],\,t_2\in[0,x_2]\\ \eta\in[0,\rho]}} \left|\int_{\mathbb{R}^n} \sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}} \prod_{i=1}^{n} |y_i|^{\ell_i} f((t_1,t_2,0,\ldots,0)\eta + y\sqrt{1-\eta^2})d\gamma_n(y)\right|.\end{aligned} \tag{34}$$

Proof. By integrating by parts, note that

$$\begin{aligned}&\frac{d}{d\rho}\int y_2 f(y\sqrt{1-\rho^2})d\gamma_n(y) \\ &= \frac{\rho}{1-\rho^2}\int y_2((n+1)-y_2^2)f(y\sqrt{1-\rho^2})d\gamma_n(y) - \sum_{i\neq 2}\frac{\rho}{1-\rho^2}\int y_i^2 y_2 f(y\sqrt{1-\rho^2})d\gamma_n(y).\end{aligned}$$

So, using $\int_{\mathbb{R}^n} y_2 f(y)d\gamma_n(y) = 0$ and the Fundamental Theorem of Calculus,

$$\begin{aligned}&\int y_2 f(y\sqrt{1-\rho^2})d\gamma_n(y) \\ &\leq \frac{\rho^2}{1-\rho^2}\sup_{\eta\in[0,\rho]}\left(\int y_2((n+1)-y_2^2)f(y\sqrt{1-\eta^2})d\gamma_n(y) - \sum_{i\neq 2}\int y_i^2 y_2 f(y\sqrt{1-\eta^2})d\gamma_n(y)\right).\end{aligned} \tag{35}$$

By integrating by parts again, note that

$$\begin{aligned}&\frac{\partial}{\partial x_2}\int y_2 f((0,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y) \\ &= \frac{\rho}{\sqrt{1-\rho^2}}\int f((0,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})(y_2^2-1)d\gamma_n(y).\end{aligned} \tag{36}$$

Applying the Fundamental Theorem of Calculus to (36) and then using (35),

$$\begin{aligned}
&\left|\int y_2 f((0,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right| \\
&\leq |x_2| \sup_{t\in[0,x_2]}\left|\frac{\rho}{\sqrt{1-\rho^2}}\int f((0,t,0,\ldots,0)\rho + y\sqrt{1-\rho^2})(y_2^2-1)d\gamma_n(y)\right| \\
&+\frac{\rho^2}{1-\rho^2}\sup_{\eta\in[0,\rho]}\left(\int y_2((n+1)-y_2^2)f(y\sqrt{1-\eta^2})d\gamma_n(y) - \sum_{i\neq 2}\int y_i^2 y_2 f(y\sqrt{1-\eta^2})d\gamma_n(y)\right).
\end{aligned} \tag{37}$$

By integrating by parts as before,

$$\begin{aligned}
&\frac{\partial}{\partial x_1}[x_2\int y_2 f((x_1,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y)] \\
&\qquad = x_2\frac{\rho}{\sqrt{1-\rho^2}}\int y_2 y_1 f((x_1,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y).
\end{aligned} \tag{38}$$

Combining (8), (37) and (38),

$$\begin{aligned}
&|(d/d\rho)T_\rho f(x)| \leq |x_1|\left|\int y_1 f((x_1,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right| \\
&+|x_2|^2\sup_{t\in[0,x_2]}\left|\frac{\rho}{\sqrt{1-\rho^2}}\int f((0,t,0,\ldots,0)\rho + y\sqrt{1-\rho^2})(y_2^2-1)d\gamma_n(y)\right| \\
&+\frac{\rho^2|x_2|}{1-\rho^2}\sup_{\eta\in[0,\rho]}\left(\int y_2((n+1)-y_2^2)f(y\sqrt{1-\eta^2})d\gamma_n(y) - \sum_{i\neq 2}\int y_i^2 y_2 f(y\sqrt{1-\eta^2})d\gamma_n(y)\right) \\
&+|x_1x_2|\sup_{t\in[0,x_1]}\frac{\rho}{\sqrt{1-\rho^2}}\left|\int y_2y_1 f((t,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right| \\
&+\frac{\rho}{\sqrt{1-\rho^2}}\left|\int(\sum_{i=1}^n(y_i^2-1))f((x_1,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right|.
\end{aligned} \tag{39}$$

We then deduce (34) from (39). □

4.1. **The Main Lemma.**

Lemma 4.4. *Fix $n \geq 2$, $k = 3$. Let $0 < \eta < \rho < e^{-20(n+1)^{10^{12}}n^3(n+2)!}$. Let $\{A_i\}_{i=1}^k$ be a partition of $\mathbb{R}^n$ such that (13) holds. Let $\Pi \subseteq \mathbb{R}^n$ be a fixed 2-dimensional plane such that $0 \in \Pi$. Assume that, for each pair $i,j \in \{1,2,\ldots,k\}$ with $i \neq j$, there exists $\lambda' > 0$ and there exists a regular simplicial conical partition $\{B'_p\}_{p=1}^k \subseteq \mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} y(1_{A_i}(y) - 1_{A_j}(y))d\gamma_n(y) = \lambda'\int_{\mathbb{R}^n} y(1_{B'_i}(y) - 1_{B'_j}(y))d\gamma_n(y), \tag{40}$$

such that

$$\int_{B'_p} x d\gamma_n(x) \in \Pi, \forall p \in \{i,j\}, \tag{41}$$

and such that

$$\begin{aligned}&\{x \in B_i' \cup B_j' \colon 1_{A_i}(x) - 1_{A_j}(x) \neq 1_{B_i'}(x) - 1_{B_j'}(x)\}\\ &\quad\subseteq \{x \in B_i' \cup B_j' \colon |d(x, (\partial B_i') \cup (\partial B_j'))| < \eta \vee ||x||_2 \geq \sqrt{-2\log\eta} + (\rho+\eta)\sqrt{-2\log\rho}\}.\end{aligned} \tag{42}$$

Then, for each pair $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, there exists $\lambda'' > 0$ and there exists a regular simplicial conical partition $\{B_p''\}_{p=1}^k \subseteq \mathbb{R}^n$ such that $\int_{\mathbb{R}^n} y(1_{A_i}(y) - 1_{A_j}(y))d\gamma_n(y) = \lambda'' \int_{\mathbb{R}^n} y(1_{B_i''}(y) - 1_{B_j''}(y))d\gamma_n(y)$, such that $\int_{B_p''} x d\gamma_n(x) \in \Pi$, $\forall\, p \in \{i, j\}$, and such that

$$\begin{aligned}&\{x \in B_i'' \cup B_j'' \colon 1_{A_i}(x) - 1_{A_j}(x) \neq 1_{B_i''}(x) - 1_{B_j''}(x)\}\\ &\quad\subseteq \{x \in B_i'' \cap B_j'' \colon |d(x, (\partial B_i'') \cup (\partial B_j''))| < \rho\eta \vee ||x||_2 \geq \sqrt{-2\log(\rho\eta)} + 1\}.\end{aligned} \tag{43}$$

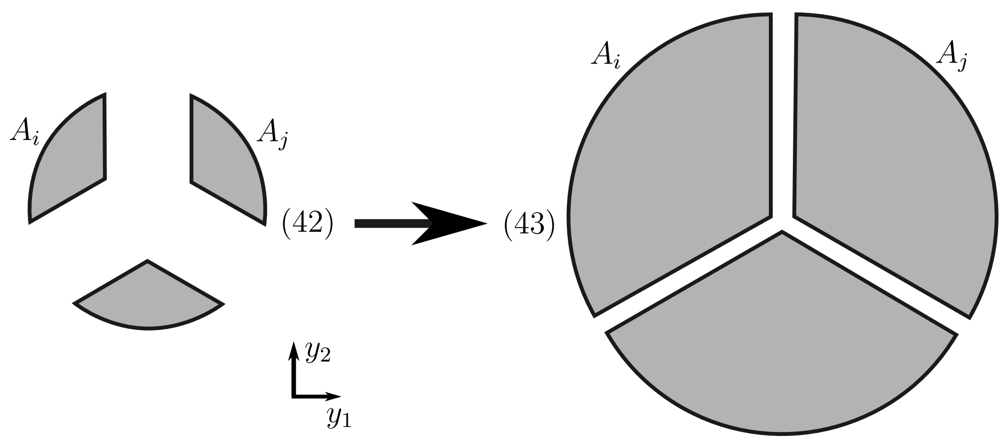

FIGURE 1. Depiction of Lemma 4.4

Proof. Fix $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. By applying a rotation to $\mathbb{R}^n$, we assume that $B_i' \cap B_j' \subseteq \{x \in \mathbb{R}^n \colon x_1 = 0\}$, and $B_i' \subseteq \{x \in \mathbb{R}^n \colon x_1 \geq 0\}$. Assume that (42) and (40) hold. Let $n_i' \in \mathbb{R}^n$ denote the interior unit normal of B_i' such that n_i' is normal to $(\partial B_i') \setminus B_j'$, and let $n_j' \in \mathbb{R}^n$ denote the interior unit normal of B_j' such that n_j' is normal to $(\partial B_j') \setminus B_i'$. Define B_i, B_j such that

$$\begin{aligned}B_i &= B_{i, \frac{2\eta}{\sqrt{-2\log\eta}}} := B_i' \cup \{x \in \mathbb{R}^n \colon x_1 \geq 0 \wedge \langle n_i', x / \, ||x||_2\rangle \geq -2\eta/\sqrt{-2\log\eta}\},\\ B_j &= B_{j, \frac{2\eta}{\sqrt{-2\log\eta}}} := B_j' \cup \{x \in \mathbb{R}^n \colon x_1 \leq 0 \wedge \langle n_j', x / \, ||x||_2\rangle \geq -2\eta/\sqrt{-2\log\eta}\}.\end{aligned} \tag{44}$$

Let $x = (x_1, \ldots, x_n) \in B_i \cup B_j$. If $x_1 < \sqrt{n}\sqrt{1-\rho^2}/\rho$, then

$$\gamma_n\left(\delta_{\frac{(B_i \cap B_j) - x\rho}{\sqrt{1-\rho^2}}}\right) \geq \frac{e^{-n/2}}{\sqrt{2\pi}} \int_{\sqrt{n}}^{\infty} e^{-t^2/2} dt / \sqrt{2\pi} \geq \frac{e^{-n/2}}{2\pi} \frac{1}{2\sqrt{n}} e^{-n/2} \geq \frac{1}{100\sqrt{n}} e^{-n}.$$

So, using Lemma 3.2, (7), and $\rho < 10^{-5} n^{-3/2} e^{-n}$, if $x \in B_i \cup B_j$ then

$$\mathrm{sign}(x_1) \cdot LT_\rho(1_{B_i} - 1_{B_j})(x) \geq \begin{cases} \frac{1}{9} |x_1| \, e^{-x_1^2\rho^2/(2(1-\rho^2))} & , |x_1| \leq 1 \vee x_2 \geq 0 \\ \frac{1}{9} \frac{|x_1|}{\rho|x_2|} e^{-||x||_2^2\rho^2/(2(1-\rho^2))} & , \rho x_2 \leq -1/\sqrt{3}. \end{cases} \tag{45}$$

Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ be a rotation such that the x_1-axis is fixed. For any such rotation, let

$$g(x) = g_\sigma(x) := 1_{A_i}(\sigma x) - 1_{A_j}(\sigma x) - (1_{B_i}(\sigma x) - 1_{B_j}(\sigma x)). \tag{46}$$

By (40), and since $B_i \cup B_j$ is symmetric with respect to reflection across $B_i \cap B_j \subseteq \{x \in \mathbb{R}^n \colon x_1 = 0\}$, $\exists\, \lambda > 0$ such that $\int y(1_{A_i}(y) - 1_{A_j}(y))d\gamma_n(y) = \lambda \int y(1_{B_i}(y) - 1_{B_j}(y))d\gamma_n(y)$. So $\int_{\mathbb{R}^n} y_2 g(y) d\gamma_n(y) = 0$, for all such rotations σ. For $x \in \mathbb{R}^n$,

$$\left|LT_\rho(1_{A_i} - 1_{A_j})(\sigma x) - LT_\rho(1_{B_i} - 1_{B_j})(\sigma x)\right| \leq \left|LT_\rho g(x)\right|. \tag{47}$$

By (46), $|g| \leq 2$. Applying (42) and (44) and the inclusion-exclusion principle, $g = 0$ on the set

$$\begin{aligned}\{y \in \mathbb{R}^n \colon\, & d(\sigma y - \rho x, (\partial B_i') \cup (\partial B_j')) > \eta + 3\eta \\ & \wedge \left|\left|\sigma y - \rho x\right|\right|_2 \leq \sqrt{-2\log\eta} + (\rho + \eta)\sqrt{-2\log\rho}\}.\end{aligned}$$

Let $x \in \mathbb{R}^n$ with $\left|\left|x\right|\right|_2^2 \leq -4\log(\eta\rho)$. Since $0 < \eta < \rho$, we have $\rho\left|\left|x\right|\right|_2 \leq -4\rho\log(\eta\rho) \leq -8\rho\log\eta$. By (42), (41) and the inclusion-exclusion principle, $g \neq 0$ only on the following sets: $\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, (\partial B_i') \cup (\partial B_j'))| \leq 4\eta/\sqrt{1-\rho^2}\}$ and $\{y \in \mathbb{R}^n \colon \left|\left|\sigma y - \rho x\right|\right|_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\}$. Then Lemma 4.2 says

$$\begin{aligned}&\sup_{\substack{t_1 \in [\min(x_1,0),\max(x_1,0)] \\ t_2 \in [\min(x_2,0),\max(x_2,0)] \\ \alpha \in [0,\rho]}} \left|\int_{\mathbb{R}^n} \sum_{\substack{\ell \in \mathbb{N}^n \colon \\ 0 \leq |\ell| \leq 3}} \prod_{i=1}^n |y_i|^{\ell_i} g((t_1, t_2, 0, \ldots, 0)\alpha + y\sqrt{1-\alpha^2}) d\gamma_n(y)\right| \\ &\qquad \leq 500000 n^3 4\eta + 200(n+2)!((-2(1-2\rho)\log\eta)^{(n+1)/2} + 1)\eta^{1-2\rho}.\end{aligned}$$

Using Lemma 4.3,

$$\begin{aligned}&\left|\left|x\right|\right|_2^2 \leq -4\log(\rho\eta) \,\wedge\, |x_1| \geq (\rho\eta)^{1/3} \\ &\Longrightarrow \left|LT_\rho(1_{A_i} - 1_{A_j})(x_1, x_2, 0, \ldots, 0) - LT_\rho(1_{B_i} - 1_{B_j})(x_1, x_2, 0, \ldots, 0)\right| \\ &\qquad \leq [|x_1| + 2\rho(|x_2|^2 + (n+1)\rho|x_2| + |x_1 x_2| + 2n)] \\ &\qquad\qquad \cdot [500000 n^3 4\eta + 200(n+2)!((-2(1-2\rho)\log\eta)^{(n+1)/2} + 1)\eta^{1-2\rho}] \\ &\qquad < 10^7 (n+2)!(-2\log\eta)^{(n+5)/2}\eta^{1-2\rho}.\end{aligned} \tag{48}$$

Also, by (45), and using that $0 < \eta < \rho < 10^{-5} n^{-3/2} e^{-n}$,

$$\begin{aligned}&\left|\left|x\right|\right|_2^2 \leq -4\log(\rho\eta) \,\wedge\, x \in B_i \cup B_j \\ &\quad\Longrightarrow \mathrm{sign}(x_1) \cdot (LT_\rho(1_{B_i} - 1_{B_j})(x)) > \begin{cases} \frac{1}{9}|x_1|(\rho\eta)^{2\rho^2/(1-\rho^2)} & , |x_1| \leq 1 \,\vee\, x_2 \geq 0 \\ \frac{1}{9}\frac{|x_1|}{\rho|x_2|}(\rho\eta)^{2\rho^2/(1-\rho^2)} & , \rho x_2 \leq -1/\sqrt{3} \end{cases}.\end{aligned} \tag{49}$$

Combining (48) and (49), using (46) and $0 < \eta < \rho < e^{-20(n+1)10^{12}(n+2)!}$,

$$\begin{aligned}&|x_1| \geq (\rho\eta)^{1/3} \wedge \left|\left|x\right|\right|_2^2 \leq -4\log(\rho\eta) \wedge x \in B_i \cup B_j \\ &\quad\Longrightarrow \left|LT_\rho(1_{A_i} - 1_{A_j})(x) - LT_\rho(1_{B_i} - 1_{B_j})(x)\right| \leq \eta^{4/5} \\ &\qquad \wedge\, \mathrm{sign}(x_1) \cdot LT_\rho(1_{B_i} - 1_{B_j})(x) \geq (\rho\eta)^{2\rho^2/(1-\rho^2)}(\rho\eta)^{1/3}\min\left(1, \frac{1}{\rho\sqrt{-4\log\eta\rho}}\right).\end{aligned} \tag{50}$$

By (50),

$$|x_1| \geq (\rho\eta)^{1/3} \wedge ||x||_2^2 \leq -4\log(\rho\eta) \wedge x \in B_i \cup B_j \Longrightarrow \mathrm{sign}(x_1)\cdot LT_\rho(1_{A_i} - 1_{A_j})(x) > 0. \quad (51)$$

Finally, applying (51) to Lemma 2.7 for all $i', j' \in \{1, \dots, k\}$, $i' \neq j'$, and using (44) together with the inclusion-exclusion principle,

$$|x_1| \geq (\rho\eta)^{1/3} \wedge ||x||_2^2 \leq -4\log(\rho\eta) \wedge x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot (1_{A_i}(x) - 1_{A_j}(x)) > 0. \quad (52)$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with $||x||_2^2 \leq -2\log\rho$, we have $\rho\, ||x||_2 \leq \rho\sqrt{-2\log\rho}$. Suppose also that $|x_1| \leq \eta$ and $x \in B_i \cup B_j$. By (44), (42), (52), (41) and the inclusion-exclusion principle, $g \neq 0$ only on the following sets:

$$\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')])| \leq \eta/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')])| \leq (\rho\eta)^{1/4} \wedge ||\sigma y - \rho x||_2 \geq \sqrt{-2\log\eta}\},$$
$$\{y \in \mathbb{R}^n \colon\ ||\sigma y - \rho x||_2 \geq (1 + 1/10)\sqrt{-3\log(\eta\rho)}/\sqrt{1-\rho^2}\}.$$

We then apply Lemma 4.2 to get

$$\sup_{\substack{t_1 \in [\min(x_1,0), \max(x_1,0)] \\ t_2 \in [\min(x_2,0), \max(x_2,0)] \\ \alpha \in [0,\rho]}} \left| \int_{\mathbb{R}^n} \sum_{\substack{\ell \in \mathbb{N}^n\colon \\ 0 \leq |\ell| \leq 3}} \prod_{i=1}^n |y_i|^{\ell_i}\, g((t_1, t_2, 0, \dots, 0)\alpha + y\sqrt{1-\alpha^2}) d\gamma_n(y) \right|$$
$$\leq 500000 n^3 \eta + 500000 n^3 (\rho\eta)^{1/4}(-2(1-\rho)^2 \log\eta + 1)\eta^{(1-\rho)^2}$$
$$+ 200(n+2)!((-3\log(\rho\eta))^{(n+1)/2} + 1)(\rho\eta)^{3/2} + 1600(n+2)!2\eta/\sqrt{-2\log\eta}.$$

So, using Lemma 4.3, and $0 < \eta < \rho < e^{-20(n+1)10^{12} n^3 (n+2)!}$,

$$\begin{aligned}
&\rho^{3/4}\eta \leq |x_1| \leq \eta \ \wedge\ ||x||_2^2 \leq -2\log\rho \ \wedge\ x \in B_i \cup B_j \\
&\Longrightarrow \left| LT_\rho(1_{A_i} - 1_{A_j})(x_1, x_2, 0, \dots, 0) - LT_\rho(1_{B_i} - 1_{B_j})(x_1, x_2, 0, \dots, 0) \right| \\
&\quad \leq [|x_1| + 2\rho(|x_2|^2 + (n+1)\rho\, |x_2| + |x_1 x_2| + 2n)] \\
&\qquad \cdot \Big[500000 n^3 3\eta + 500000 n^3 (\rho\eta)^{1/4}(-2(1-\rho)^2\log\eta + 1)\eta^{(1-\rho)^2} \\
&\qquad\quad + 200(n+2)!((-3\log(\rho\eta))^{(n+1)/2} + 1)(\rho\eta)^{3/2} + 1600(n+2)!2\eta/\sqrt{-2\log\eta} \Big] \\
&\quad < \frac{1}{10}\eta\rho^{3/4}.
\end{aligned} \quad (53)$$

Also, by (45),

$$\eta\rho^{3/4} \leq |x_1| \leq \eta \wedge ||x||_2^2 \leq -2\log\rho \wedge x \in B_i \cup B_j \Longrightarrow \mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i} - 1_{B_j})(x) > \frac{1}{10}|x_1|. \quad (54)$$

Combining (53) and (54), and using (46),

$$\eta\rho^{3/4} \leq |x_1| \leq \eta \wedge ||x||_2^2 \leq -2\log\rho \wedge x \in B_i \cup B_j \Longrightarrow \mathrm{sign}(x_1)\cdot LT_\rho(1_{A_i} - 1_{A_j})(x) > 0. \quad (55)$$

Similarly, by (53) and (46), we have the following estimate.

$$\eta \leq |x_1| \leq (\rho\eta)^{1/3} \ \wedge\ ||x||_2^2 \leq 1 \ \wedge\ x \in B_i \cup B_j \Longrightarrow \mathrm{sign}(x_1)\cdot LT_\rho(1_{A_i} - 1_{A_j})(x) > 0. \quad (56)$$

Finally, applying (55) to Lemma 2.7 for all $i', j' \in \{1, \ldots, k\}$, and using (44) together with the inclusion-exclusion principle, (51) and (56),

$$\eta\rho^{3/4} \leq |x_1| \leq \eta \wedge ||x||_2^2 \leq -2\log\rho \wedge x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1) \cdot (1_{A_i}(x) - 1_{A_j}(x)) > 0. \quad (57)$$

In summary, (57) and (52) improve our initial assumption (42). We now repeat the above procedure with the improved assumptions. Before continuing, we need to redefine B_i, B_j. Via (44), let

$$B_i := B_{i,\min\left(\frac{2\rho^{3/4}\eta}{\sqrt{-2\log\rho}}, \frac{2\eta}{\sqrt{-2\log\eta}}\right)}, \; B_j := B_{j,\min\left(\frac{2\rho^{3/4}\eta}{\sqrt{-2\log\rho}}, \frac{2\eta}{\sqrt{-2\log\eta}}\right)} \quad (58)$$

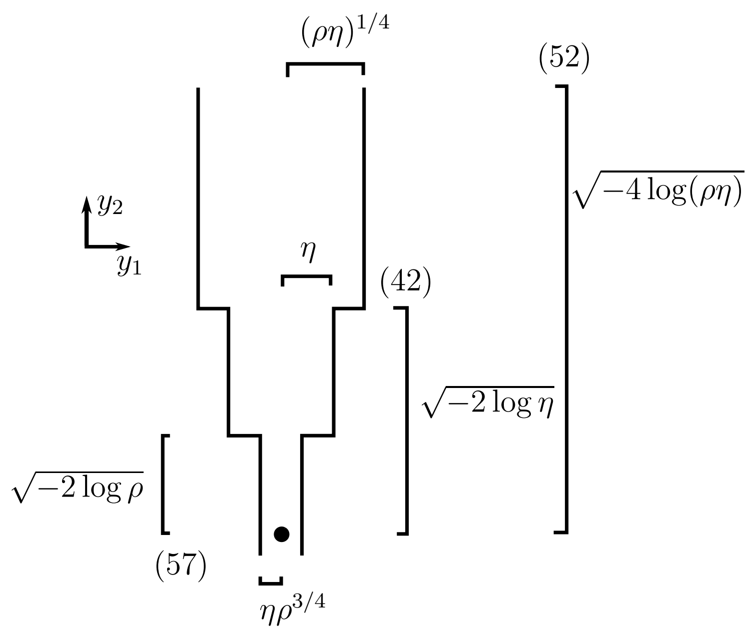

FIGURE 2. Integration regions near $B_i' \cap B_j'$ for (59).

Let x with $||x||_2^2 \leq -4\log\rho$, $\eta\rho \leq |x_1| \leq \eta$. Let $B := (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')]$. Suppose $x \in B_i \cup B_j$ also. By (57), (42), (52), (41) and the inclusion-exclusion principle, $g \neq 0$ only on the following sets

$$\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq \eta\rho^{3/4}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq \eta/\sqrt{1-\rho^2}, ||\sigma y - \rho x||_2 > \sqrt{-2\log\rho}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq (\rho\eta)^{1/4}/\sqrt{1-\rho^2}, ||\sigma y - \rho x||_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n \colon \; ||\sigma y - \rho x||_2 > (1 + 1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.$$

We then apply Lemma 4.2 to get

$$\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}}\left|\int_{\mathbb{R}^n}\sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}}\prod_{i=1}^{n}|y_i|^{\ell_i}\, g((t_1,t_2,0,\ldots,0)\alpha+y\sqrt{1-\alpha^2})d\gamma_n(y)\right|$$

$$\begin{aligned}&\leq \eta\rho^{3/4}500000n^3+500000n^3\eta(-2(1-\sqrt{2}\rho)^2\log\rho+1)\rho^{(1-\sqrt{2}\rho)^2}\\ &\quad+500000n^3(\eta\rho)^{1/4}(-2(1-\sqrt{2}\rho)^2\log\eta+1)\eta^{(1-\sqrt{2}\rho)^2}\\ &\quad+200(n+2)!((-3\log(\eta\rho))^{(n+1)/2}+1)(\rho\eta)^{3/2}\\ &\quad+1600(n+2)!\min(2\rho^{3/4}\eta/\sqrt{-2\log\rho},2\eta/\sqrt{-2\log\eta}).\end{aligned}\tag{59}$$

Applying (59) to Lemma 4.3, using (46) and $\eta<\rho<e^{-20(n+1)10^{12}n^3(n+2)!}$,

$$\begin{aligned}&\|x\|_2^2\leq-4\log(\rho)\ \wedge\ \eta\rho\leq|x_1|\leq\eta\ \wedge x\in B_i\cup B_j\\ &\qquad\Longrightarrow\left|LT_\rho(1_{A_i}-1_{A_j})(x)-LT_\rho(1_{B_i}-1_{B_j})(x)\right|<\frac{1}{10}\rho\eta.\end{aligned}\tag{60}$$

Also, by (45),

$$\rho\eta\leq|x_1|\leq\eta\wedge\|x\|_2^2\leq-4\log\rho\wedge x\in B_i\cup B_j\Longrightarrow\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x)>\frac{1}{10}\rho\eta.\tag{61}$$

So, combining (60), (61) for all $i',j'\in\{1,\ldots,k\}$, $i'\neq j'$, Lemma 2.7, (58), and by applying the inclusion-exclusion principle, (51) and (56),

$$\rho\eta\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq-4\log\rho\ \wedge\ x\in B_i'\cup B_j'\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\tag{62}$$

The estimate (62) now has a cascading effect on the estimates below. From (62),

$$\rho^{.9}\eta\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq-4\log\rho\ \wedge\ x\in B_i'\cup B_j'\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.$$

This estimate can be iterated on itself. Let $K\in\mathbb{N}$, $K\geq 1$, and let $M\in\mathbb{N}$ with $0\leq M\leq\sqrt{K}$. Suppose $\rho^{.9K}>\eta^{1/5}$. We prove by induction on K and M that

$$\begin{aligned}&2^{M^2}\eta\rho^{.9K}\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq-2^{M+2}\log\rho\ \wedge\ x\in B_i'\cup B_j'\\ &\qquad\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\end{aligned}\tag{63}$$

We already verified the case $M=0,K=1$. We assume that, for $0\leq m<M$,

$$\begin{aligned}&2^{m^2}\eta\rho^{.9K}\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq-2^{m+2}\log\rho\ \wedge\ x\in B_i'\cup B_j'\\ &\qquad\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\end{aligned}\tag{64}$$

Assume also that, for $M\leq m\leq\sqrt{K-1}$ and $K\geq 1$,

$$\begin{aligned}&2^{m^2}\eta\rho^{.9(K-1)}\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq-2^{m+2}\log\rho\ \wedge\ x\in B_i'\cup B_j'\\ &\qquad\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\end{aligned}\tag{65}$$

We will conclude that (64) holds for $m=M$, i.e.

$$\begin{aligned}&2^{M^2}\eta\rho^{.9K}\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq-2^{M+2}\log\rho\ \wedge\ x\in B_i'\cup B_j'\\ &\qquad\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\end{aligned}\tag{66}$$

We repeat the calculations (58) through (62). Redefine B_i,B_j so that

$$B_i := B_{i,\min\left(\frac{2\eta\rho^{.9K}}{\sqrt{-4\log\rho}},\frac{2\eta}{\sqrt{-2\log\eta}}\right)},\quad B_j := B_{j,\min\left(\frac{2\eta\rho^{.9K}}{\sqrt{-4\log\rho}},\frac{2\eta}{\sqrt{-2\log\eta}}\right)}. \tag{67}$$

If $M > 0$, we use (64) for $m = M-1$. For any $M \geq 0$, we use (65) for $M \leq m \leq \sqrt{K}$. Let x, M with $\|x\|_2^2 \leq -2^{M+2}\log\rho \leq -2^{\lfloor\sqrt{K}\rfloor+2}\log\rho \leq -4\log\eta$, $2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta$, $x \in B_i \cup B_j$. Let $B := (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')]$. Combining (64), (65), (42), (52), (41) and the inclusion-exclusion principle, $g \neq 0$ only on the following sets

$$\begin{aligned}
&\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq \min(M,1)\cdot 2^{(M-1)^2}\eta\rho^{.9K}/\sqrt{1-\rho^2}\},\\
&\cup_{M\leq m\leq\lfloor\sqrt{K-1}\rfloor}\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq 2^{m^2}\eta\rho^{.9(K-1)}/\sqrt{1-\rho^2},\\
&\qquad\qquad \|\sigma y - \rho x\|_2 > \min(m,1)\cdot\sqrt{-2^{m+1}\log\rho}/\sqrt{1-\rho^2}\},\\
&\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq \eta, \|\sigma y - \rho x\|_2 > \sqrt{-2^{\lfloor\sqrt{K-1}\rfloor+2}\log\rho}/\sqrt{1-\rho^2}\},\\
&\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq (\rho\eta)^{1/4}/\sqrt{1-\rho^2}, \|\sigma y - \rho x\|_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},\\
&\{y \in \mathbb{R}^n \colon \|\sigma y - \rho x\|_2 > (1+1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.
\end{aligned}$$

We then apply Lemma 4.2 to get

$$\begin{aligned}
&\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}}\left|\int_{\mathbb{R}^n}\sum_{\substack{\ell\in\mathbb{N}^n\colon\\ 0\leq|\ell|\leq 3}}\prod_{i=1}^n |y_i|^{\ell_i}\, g((t_1,t_2,0,\ldots,0)\alpha + y\sqrt{1-\alpha^2})d\gamma_n(y)\right|\\
&\quad\leq \min(M,1)\cdot 2^{(M-1)^2}\eta\rho^{.9K}500000n^3\\
&\qquad + \eta(-(1-\rho)^2 2^{\lfloor\sqrt{K-1}\rfloor+2}\log\rho+1)\rho^{(1-\rho)^2 2^{\lfloor\sqrt{K-1}\rfloor+1}}500000n^3\\
&\qquad + 500000n^3\sum_{m=M}^{\lfloor\sqrt{K-1}\rfloor}\eta\rho^{.9(K-1)}(-(1-\rho)^2 2^{m+1}\log\rho+1)2^{m^2}\rho^{(1-\rho)^2 2^m\cdot\min(m,1)}\\
&\qquad + 500000n^3(\eta\rho)^{1/4}(-2(1-\sqrt{2}\rho)^2\log\eta+1)\eta^{(1-\sqrt{2}\rho)^2}\\
&\qquad + 200(n+2)!((-3\log(\eta\rho))^{(n+1)/2}+1)(\rho\eta)^{3/2}\\
&\qquad + 1600(n+2)!\min(2\eta\rho^{.9K}/\sqrt{-4\log\rho}, 2\eta/\sqrt{-2\log\eta}).
\end{aligned} \tag{68}$$

Applying (68) to Lemma 4.3, using (46), $\eta < \rho < e^{-20(n+1)10^{12}n^3(n+2)!}$, and $\rho^{.9K} > \eta^{1/5}$,

$$\begin{aligned}
&\|x\|_2^2 \leq -2^{M+2}\log(\rho) \,\wedge\, 2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \,\wedge\, x \in B_i \cup B_j\\
&\qquad\Longrightarrow \left|LT_\rho(1_{A_i} - 1_{A_j})(x) - LT_\rho(1_{B_i} - 1_{B_j})(x)\right| < \frac{1}{10}2^{M^2}\eta\rho^{.9K}.
\end{aligned} \tag{69}$$

Also, by (45),

$$\begin{aligned}
&2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{M+2}\log\rho \,\wedge\, x \in B_i \cup B_j\\
&\qquad\Longrightarrow \mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i} - 1_{B_j})(x) > \frac{1}{10}2^{M^2}\eta\rho^{.9K}.
\end{aligned} \tag{70}$$

So, combining (69), (70) for all $i', j' \in \{1, \ldots, k\}$, $i' \neq j'$, Lemma 2.7, (67), and by applying the inclusion-exclusion principle, (51) and (56),

$$2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \wedge ||x||_2^2 \leq -2^{M+2}\log\rho \wedge x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \tag{71}$$

Thus, the inductive step is completed.

Let $K \in \mathbb{N}$ with $-2\log\eta \leq -2^{\lfloor\sqrt{K}\rfloor+2}\log\rho \leq -4\log\eta$. Then (71) and (42) say that

$$2^K\eta\rho^{.9K} \leq |x_1| \leq 1 \wedge ||x||_2^2 \leq -2\log\eta \wedge x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \tag{72}$$

We perform another induction, though this time we hold K fixed and use the additional ingredient (72). Let $M, R \in \mathbb{N}$ with $0 \leq M \leq \sqrt{K}$, $R \geq 0$ such that $\rho^{.9(K+R)} > \eta^{1/5}$. We will induct on M and R. We assume that, for $0 \leq m < M$,

$$\begin{aligned} 2^{m^2}\eta\rho^{.9(K+R)} \leq |x_1| \leq \eta \,\wedge\, ||x||_2^2 \leq -2^{m+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{73}$$

We know that the case $R = 0, 0 \leq M \leq \sqrt{K}$ of (73) holds by (66). We therefore assume that $R \geq 1$. Assume also that, for $M \leq m \leq \sqrt{K}$,

$$\begin{aligned} 2^{m^2}\eta\rho^{.9(K+R-1)} \leq |x_1| \leq \eta \,\wedge\, ||x||_2^2 \leq -2^{m+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{74}$$

We will conclude that (73) holds for $m = M$, i.e.

$$\begin{aligned} 2^{M^2}\eta\rho^{.9(K+R)} \leq |x_1| \leq \eta \,\wedge\, ||x||_2^2 \leq -2^{M+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{75}$$

Redefine B_i, B_j so that

$$B_i := B_{i,\min\left(\frac{2\eta\rho^{.9(K+R)}}{\sqrt{-4\log\rho}}, \frac{2\eta}{\sqrt{-2\log\eta}}\right)}, \quad B_j := B_{j,\min\left(\frac{2\eta\rho^{.9(K+R)}}{\sqrt{-4\log\rho}}, \frac{2\eta}{\sqrt{-2\log\eta}}\right)}. \tag{76}$$

If $M > 0$, we use (73) for $m = M - 1$. For any $M \geq 0$, we also use (74) for $M \leq m \leq \sqrt{K}$. Let x, M with $||x||_2^2 \leq -2^{M+2}\log\rho \leq -2^{\lfloor\sqrt{K}\rfloor+2}\log\rho \leq -4\log\eta$, $2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta$, $x \in B_i \cup B_j$. Let $B := (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')]$. Combining (73), (74), (42), (52), (41), and the fact that $-2\log\eta \leq -2^{\lfloor\sqrt{K}\rfloor+2}\log\rho \leq -4\log\eta$, we conclude that $g \neq 0$ only on the following sets:

$$\begin{aligned}
&\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq \min(M,1)\cdot 2^{(M-1)^2}\eta\rho^{.9(K+R)}/\sqrt{1-\rho^2}\},\\
&\cup_{M \leq m \leq \lfloor\sqrt{K}\rfloor} \{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq 2^{m^2}\eta\rho^{.9(K+R-1)}/\sqrt{1-\rho^2},\\
&\qquad\qquad\qquad ||\sigma y - \rho x||_2 > \min(m,1)\cdot\sqrt{-2^{m+1}\log\rho}/\sqrt{1-\rho^2}\},\\
&\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq (\rho\eta)^{1/4}/\sqrt{1-\rho^2}, ||\sigma y - \rho x||_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},\\
&\{y \in \mathbb{R}^n \colon\ ||\sigma y - \rho x||_2 > (1 + 1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.
\end{aligned}$$

We then apply Lemma 4.2 to get

$$\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}} \left| \int_{\mathbb{R}^n} \sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}} \prod_{i=1}^{n} |y_i|^{\ell_i}\, g((t_1,t_2,0,\ldots,0)\alpha + y\sqrt{1-\alpha^2})d\gamma_n(y)\right|$$
$$\begin{aligned}&\leq \min(M,1)\cdot 2^{(M-1)^2}\eta\rho^{.9(K+R)}500000n^3\\ &\quad + 500000n^3\sum_{m=M}^{\lfloor\sqrt{K}\rfloor}\eta\rho^{.9(K+R-1)}(-(1-\rho)^2 2^{m+1}\log\rho+1)2^{m^2}\rho^{(1-\rho)^2 2^m}\\ &\quad + 500000n^3(\eta\rho)^{1/4}(-2(1-\sqrt{2}\rho)^2\log\eta+1)\eta^{(1-\sqrt{2}\rho)^2}\\ &\quad + 200(n+2)!((-3\log(\eta\rho))^{(n+1)/2}+1)(\rho\eta)^{3/2}\\ &\quad + 1600(n+2)!\min(2\eta\rho^{.9(K+R)}/\sqrt{-4\log\rho}, 2\eta/\sqrt{-2\log\eta}).\end{aligned} \tag{77}$$

Applying (77) to Lemma 4.3, using (46), $\eta<\rho<e^{-20(n+1)10^{12}n^3(n+2)!}$, and $\rho^{.9(K+R)}>\eta^{1/5}$,

$$\begin{aligned}&||x||_2^2\leq -2^{M+2}\log(\rho)\,\wedge\, 2^{M^2}\eta\rho^{.9(K+R)}\leq|x_1|\leq\eta\,\wedge\, x\in B_i\cup B_j\\ &\qquad\Longrightarrow \left|LT_\rho(1_{A_i}-1_{A_j})(x)-LT_\rho(1_{B_i}-1_{B_j})(x)\right|<\frac{1}{10}2^{M^2}\eta\rho^{.9(K+R)}.\end{aligned}\tag{78}$$

Also, by (45),

$$\begin{aligned}&2^{M^2}\eta\rho^{.9(K+R)}\leq|x_1|\leq\eta\,\wedge\,||x||_2^2\leq-2^{M+2}\log\rho\,\wedge\, x\in B_i\cup B_j\\ &\qquad\Longrightarrow \mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x)>\frac{1}{10}2^{M^2}\eta\rho^{.9(K+R)}.\end{aligned}\tag{79}$$

So, combining (78), (79) for all $i',j'\in\{1,\ldots,k\}$, $i'\neq j'$, Lemma 2.7, (76), and by applying the inclusion-exclusion principle, (51) and (56),

$$\begin{aligned}&2^{M^2}\eta\rho^{.9(K+R)}\leq|x_1|\leq\eta\,\wedge\,||x||_2^2\leq-2^{M+2}\log\rho\,\wedge\, x\in B_i'\cup B_j'\\ &\qquad\Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\end{aligned}\tag{80}$$

Thus, the inductive step is completed. Let $M=\lfloor\sqrt{K}\rfloor$. Let $R\in\mathbb{N}$ such that $\eta^{1/5}\leq\rho^{.9(K+R)}\leq\eta^{1/5}\rho^{-.9}$. If no such R exists, then $\rho^{.9K}<\eta^{1/5}$, so $\rho^{.45K}<\eta^{1/10}$, and (82) below holds by combining (72) and (42). Otherwise, $R\geq 0$, so (80) and (42) say that

$$2^K\eta^{6/5}\rho^{-.9}\leq|x_1|\leq 1\,\wedge\,||x||_2^2\leq-2\log\eta\,\wedge\,x\in B_i'\cup B_j'\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\tag{81}$$

Since $\eta^{1/5}\leq\rho^{.9(K+R)}\leq\eta^{1/5}\rho^{-.9}$, note that $\eta^{1/10}\leq\rho^{.45(K+R)}\leq\eta^{1/10}\rho^{-.45}$, so for $R\geq 2$, we have $2^K\eta^{6/5}\rho^{-.9}\leq 2^K\rho^{.45K}\rho^{.45R}\eta^{11/10}\rho^{-.9}<\eta^{11/10}$. If $R=1$, and if $K\geq 2$, note that $2^K\eta^{6/5}\rho^{-.9}\leq 2^K\rho^{.2K}\rho^{.25K}\rho^{.45}\eta^{11/10}\rho^{-.9}<\eta^{11/10}$. If $R=0$, $K\geq 3$ then $2^K\eta^{6/5}\rho^{-.9}\leq 2^K\rho^{.1K}\rho^{.35K}\eta^{11/10}\rho^{-.9}<\eta^{11/10}$. If $1\leq R+K\leq 3$, then $(1/5)\log\eta\leq 3\log\rho$ and $2\log\rho\leq(1/5)\log\eta$, so (68) directly implies (82). More specifically, by (68), Lemma 4.3,(51) and (56), $\mathrm{sign}(x_1)\cdot(1_{A_i}-1_{A_j})(x)>0$ for $x\in B_i'\cup B_j'$ with $||x||_2^2\leq-2\log\eta$ and $\eta\rho^{.9K}\rho^{.9}\leq|x_1|\leq\eta$. Now, $\rho^{.45(K+R)}\leq\eta^{1/10}\rho^{-.45}$, so $\eta\rho^{.9K}\rho^{.9}=\eta\rho^{.45K}\rho^{.45K}\rho^{.9}\leq\eta\rho^{.45K}\rho^{.45(K+R)}\leq\eta^{11/10}$.

In the latter case, (82) follows, and in the former cases, (81) implies

$$\eta^{11/10}\leq|x_1|\leq 1\wedge||x||_2^2\leq-2\log\eta\wedge x\in B_i'\cup B_j'\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\tag{82}$$

In all cases, (82) holds. We can finally use (82) to conclude the proof.

Redefine B_i, B_j so that

$$B_i := B_{i,2\eta^{11/10}/\sqrt{-2\log\eta}}, \quad B_j := B_{j,2\eta^{11/10}/\sqrt{-2\log\eta}}. \tag{83}$$

Let x with $||x||_2^2 \leq -4\log(\eta\rho) \leq -8\log\eta$ and $\eta^{21/20}\rho^{1/2} \leq |x_1| \leq (\eta\rho)^{1/4}$. Let $B := (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')]$. Suppose $x \in B_i \cup B_j$ also. Combining (82), (52), and (41), $g \neq 0$ only on the following sets:

$$\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq \eta^{11/10}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq (\rho\eta)^{1/4}/\sqrt{1-\rho^2}, ||\sigma y - \rho x||_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n \colon\ ||\sigma y - \rho x||_2 > (1+1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.$$

We then apply Lemma 4.2 to get

$$\begin{aligned}
&\sup_{\substack{t_1 \in [\min(x_1,0),\max(x_1,0)] \\ t_2 \in [\min(x_2,0),\max(x_2,0)] \\ \alpha \in [0,\rho]}} \left| \int_{\mathbb{R}^n} \sum_{\substack{\ell \in \mathbb{N}^n \colon \\ 0 \leq |\ell| \leq 3}} \prod_{i=1}^n |y_i|^{\ell_i} g((t_1, t_2, 0, \ldots, 0)\alpha + y\sqrt{1-\alpha^2}) d\gamma_n(y) \right| \\
&\quad \leq 500000n^3\eta^{11/10} + 500000n^3(\rho\eta)^{1/4}(-2(1-2\rho)^2\log\eta + 1)\eta^{(1-2\rho)^2} \\
&\qquad + 200(n+2)!((-3\log(\eta\rho))^{(n+1)/2} + 1)(\rho\eta)^{3/2} + 1600(n+2)!2\eta^{11/10}/\sqrt{-2\log\eta}.
\end{aligned} \tag{84}$$

Applying (84) to Lemma 4.3, using (46) and $\eta < \rho < e^{-20(n+1)^{10^{12}n^3(n+2)!}}$,

$$\begin{aligned}
&||x||_2^2 \leq -4\log(\eta\rho) \ \wedge\ \eta^{21/20}\rho^{1/2} \leq |x_1| \leq (\eta\rho)^{1/4} \ \wedge\ x \in B_i \cup B_j \\
&\quad \Longrightarrow \left| LT_\rho(1_{A_i} - 1_{A_j})(x) - LT_\rho(1_{B_i} - 1_{B_j})(x) \right| < \frac{1}{10}\eta^{21/20}\rho^{1/2}.
\end{aligned} \tag{85}$$

Also, by (45),

$$\begin{aligned}
&\eta^{21/20}\rho^{1/2} \leq |x_1| \leq (\eta\rho)^{1/4} \ \wedge\ ||x||_2^2 - 4\log(\eta\rho) \ \wedge\ x \in B_i \cup B_j \\
&\quad \Longrightarrow \mathrm{sign}(x_1) \cdot LT_\rho(1_{B_i} - 1_{B_j})(x) > \frac{1}{10}\eta^{21/20}\rho^{1/2}.
\end{aligned} \tag{86}$$

So, combining (85), (86) for all $i', j' \in \{1, \ldots, k\}$, $i' \neq j'$, Lemma 2.7, (83), and by applying the inclusion-exclusion principle, (51) and (56),

$$\begin{aligned}
&\eta^{21/20}\rho^{1/2} \leq |x_1| \leq (\eta\rho)^{1/4} \ \wedge\ ||x||_2^2 \leq -4\log(\eta\rho) \ \wedge\ x \in B_i' \cup B_j' \\
&\quad \Longrightarrow \mathrm{sign}(x_1) \cdot (1_{A_i}(x) - 1_{A_j}(x)) > 0.
\end{aligned} \tag{87}$$

So, (87) and (52) say that

$$\eta^{21/20}\rho^{1/2} \leq |x_1| \wedge ||x||_2^2 \leq -4\log(\eta\rho) \wedge x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1) \cdot (1_{A_i}(x) - 1_{A_j}(x)) > 0. \tag{88}$$

Finally, we use (88) in place of (82) and repeat the computations (84) through (87) to get

$$\eta\rho \leq |x_1| \ \wedge\ ||x||_2^2 \leq -4\log(\eta\rho) \ \wedge\ x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1) \cdot (1_{A_i}(x) - 1_{A_j}(x)) > 0. \tag{89}$$

In conclusion, (43) follows from (89) and (40), letting $B_i'' := B_i'$ and $B_j'' := B_j'$.

For completeness, we derive (89). Redefine B_i, B_j so that

$$B_i := B_{i,2\eta^{21/20}\rho^{1/2}/\sqrt{-4\log(\eta\rho)}}, \quad B_j := B_{j,2\eta^{21/20}\rho^{1/2}/\sqrt{-4\log(\eta\rho)}}. \tag{90}$$

Let x with $||x||_2^2 \leq -4\log(\eta\rho) \leq -8\log\eta$ and $\eta\rho \leq |x_1| \leq (\eta\rho)^{1/3}$. Let $B := (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')]$. Suppose $x \in B_i \cup B_j$ also. Combining (88), (52), and (41), $g \neq 0$ only on the following sets:

$$\{y \in \mathbb{R}^n\colon |d(\sigma y - \rho x, B)| \leq \eta^{21/20}\rho^{1/2}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n\colon |d(\sigma y - \rho x, B)| \leq (\rho\eta)^{1/3}/\sqrt{1-\rho^2}, ||\sigma y - \rho x||_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n\colon\ ||\sigma y - \rho x||_2 > (1+1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.$$

We then apply Lemma 4.2 to get

$$\begin{aligned}&\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}} \left| \int_{\mathbb{R}^n} \sum_{\substack{\ell\in\mathbb{N}^n\colon\\ 0\leq|\ell|\leq 3}} \prod_{i=1}^n |y_i|^{\ell_i}\, g((t_1,t_2,0,\ldots,0)\alpha + y\sqrt{1-\alpha^2})d\gamma_n(y)\right|\\
&\quad\leq 500000n^3\eta^{21/20}\rho^{1/2} + 500000n^3(\rho\eta)^{1/3}(-2(1-2\rho)^2\log\eta+1)\eta^{(1-2\rho)^2}\\
&\qquad + 200(n+2)!((-3\log(\eta\rho))^{(n+1)/2}+1)(\rho\eta)^{3/2} + 1600(n+2)!2\eta^{6/5}\rho^{1/2}/\sqrt{-4\log(\eta\rho)}.\end{aligned} \tag{91}$$

Applying (91) to Lemma 4.3, using (46) and $\eta < \rho < e^{-20(n+1)^{10^{12}n^3(n+2)!}}$,

$$\begin{aligned}&||x||_2^2 \leq -4\log(\eta\rho)\ \wedge\ \eta\rho \leq |x_1| \leq (\eta\rho)^{1/3}\ \wedge\ x \in B_i \cup B_j\\
&\qquad\Longrightarrow \left|LT_\rho(1_{A_i} - 1_{A_j})(x) - LT_\rho(1_{B_i} - 1_{B_j})(x)\right| < \frac{1}{10}\eta\rho.\end{aligned} \tag{92}$$

Also, by (45),

$$\begin{aligned}&\eta\rho \leq |x_1| \leq (\eta\rho)^{1/3}\ \wedge\ ||x||_2^2 - 4\log(\eta\rho)\ \wedge\ x \in B_i \cup B_j\\
&\qquad\Longrightarrow \mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i} - 1_{B_j})(x) > \frac{1}{10}\eta\rho.\end{aligned} \tag{93}$$

So, combining (92), (93) for all $i', j' \in \{1,\ldots,k\}$, $i' \neq j'$, Lemma 2.7, (90), and by applying the inclusion-exclusion principle, (51) and (56),

$$\eta\rho \leq |x_1| \leq (\eta\rho)^{1/3}\ \wedge\ ||x||_2^2 \leq -4\log(\eta\rho)\ \wedge\ x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \tag{94}$$

Then, (94) and (52) say that

$$\eta\rho \leq |x_1|\ \wedge\ ||x||_2^2 \leq -4\log(\eta\rho)\ \wedge\ x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \tag{95}$$

Finally, (89) follows from (95), completing the proof. □

5. Proof of the Main Theorem

Theorem 5.1. *Fix $k = 3$, $n \geq 2$. Define $\Delta_k(\gamma_n)$ as in Definition 2.1 and define ψ_ρ as in (14). Let $\{B_i\}_{i=1}^k \subseteq \mathbb{R}^n$ be a regular simplicial conical partition. Then there exists $\rho_0 = \rho_0(n,k) > 0$ such that, for $\rho \in (0,\rho_0)$, $(1_{B_1},\ldots,1_{B_k})$ uniquely achieves the following supremum*

$$\sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)} \sum_{i=1}^k \int f_i LT_\rho f_i d\gamma_n = \sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)} \psi_\rho(f_1,\ldots,f_k).$$

Proof. By Lemma 2.7, let $\{A_i\}_{i=1}^k$ be a partition of $\mathbb{R}^n$ such that

$$\psi_\rho(1_{A_1},\ldots,1_{A_k}) = \sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)} \psi_\rho(f_1,\ldots,f_k). \tag{96}$$

By (10), write

$$\sum_{i=1}^k \int 1_{A_i} LT_\rho 1_{A_i} d\gamma_n = \sum_{i=1}^k \sum_{\ell\in\mathbb{N}^n} |\ell| \left|\int 1_{A_i}\sqrt{\ell!}h_\ell d\gamma_n\right|^2 \rho^{|\ell|-1}.$$

For $i\in\{1,\ldots,k\}$, let $z_i := \int_{A_i} x d\gamma_n(x) \in \mathbb{R}^n$. Then

$$\left|\sum_{i=1}^k \int 1_{A_i} LT_\rho 1_{A_i} d\gamma_n - \sum_{i=1}^k ||z_i||_{\ell_2^n}^2\right| \leq 3k\rho. \tag{97}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^k ||z_i||_{\ell_2^n}^2 &\overset{(11)}{=} \psi_0(1_{A_1},\ldots,1_{A_k}) \overset{(97)}{\geq} \psi_\rho(1_{A_1},\ldots,1_{A_k}) - 3k\rho \overset{(96)}{\geq} \psi_\rho(1_{B_1},\ldots,1_{B_k}) - 3k\rho \\ &\overset{(97)}{\geq} \psi_0(1_{B_1},\ldots,1_{B_k}) - 6k\rho \overset{(\text{Lemma } 2.5)}{=} \sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)} \psi_0(f_1,\ldots,f_k) - 6k\rho.\end{aligned}$$

For $i\in\{1,\ldots,k\}$, let $w_i := \int_{B_i} x d\gamma_n(x)$. By Lemma 3.3, let $6k\rho < \varepsilon_1$ so that

$$d_2(\{A_i\}_{i=1}^k, \{B_i\}_{i=1}^k) < 6(6k\rho)^{1/8}, \tag{98}$$

$$\inf_{\rho\in SO(n)} \left(\sum_{i=1}^k ||\rho z_i - w_i||_{\ell_2^n}^2\right)^{1/2} < 6(6k\rho)^{1/8}. \tag{99}$$

Let $x = (x_1,\ldots,x_n)\in\mathbb{R}^n$, let $i,j\in\{1,\ldots,k\}$, $i\neq j$, and write the following equality of L_2 functions

$$1_{A_i}(x) - 1_{A_j}(x) =: \sum_{\ell\in\mathbb{N}^n} c_\ell h_\ell(x)\sqrt{\ell!}. \tag{100}$$

Let $\ell = (\ell_1,\ldots,\ell_n)\in\mathbb{N}^n$. By applying an orthogonal change of coordinates to $\{A_p\}_{p=1}^k$, we may assume that $c_\ell = 0$ when $|\ell| = 1$, $\ell_1 = 0$. By (7) and (5), write

$$LT_\rho(1_{A_i} - 1_{A_j})(x) = \sum_{\ell\in\mathbb{N}^n} c_\ell\, |\ell|\, \rho^{|\ell|-1} h_\ell(x)\sqrt{\ell!}. \tag{101}$$

Let $x\in\mathbb{R}^n$ with $||x||_2^2 \leq -\log\rho^3$. Since $d_2(\{A_p\}_{p=1}^k, \{B_p\}_{p=1}^k) < 6(6k\rho)^{1/8}$, there exists $\{B_p''\}_{p=1}^k$ a regular simplicial conical partition, such that $(\sum_{p=1}^k ||1_{A_p} - 1_{B_p''}||_{L_2(\gamma_n)}^2)^{1/2} < 6(6k\rho)^{1/8}$. In particular,

$$\left|\left|\int_{\mathbb{R}^n} x(1_{A_i}(x) - 1_{A_j}(x) - (1_{B_i''}(x) - 1_{B_j''}(x)))d\gamma_n(x)\right|\right|_{\ell_2^n} < 6(6k\rho)^{1/8}, \tag{102}$$

$$\left|\left|\int_{\mathbb{R}^n} x(1_{(A_i\cup A_j)^c}(x) - 1_{(B_i''\cup B_j'')^c}(x))d\gamma_n(x)\right|\right|_{\ell_2^n} < 6(6k\rho)^{1/8}. \tag{103}$$

Since $k=3$, and since $\sum_{p=1}^{k}\int_{A_p} x d\gamma_n(x) = \int_{\mathbb{R}^n} x d\gamma_n(x) = 0$, there exists a 2-dimensional plane $\Pi \subseteq \mathbb{R}^n$ such that $0 \in \Pi$ and such that, for all $p \in \{1,\ldots,k\}$, $\int_{A_p} x d\gamma_n(x) \in \Pi$.

Let $\{B_p'\}_{p=1}^{k}$ be a regular simplicial conical partition such that

$$\left(\sum_{p=1}^{k} \|1_{B_p'} - 1_{B_p''}\|_{L_2(\gamma_n)}^2\right)^{1/2} < 10(6k\rho)^{1/16}, \tag{104}$$

such that for fixed $i \neq j$, $i,j \in \{1,\ldots,k\}$ and for some $\lambda' \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} x(1_{A_i}(x) - 1_{A_j}(x)) d\gamma_n(x) = \lambda' \int_{\mathbb{R}^n} x(1_{B_i'}(x) - 1_{B_j'}(x)) d\gamma_n(x), \tag{105}$$

and such that

$$\int_{\mathbb{R}^n} x(1_{(B_i' \cup B_j')^c}) d\gamma_n(x) \in \Pi. \tag{106}$$

Such $\{B_p'\}_{p=1}^{k}$ exists by (102), using $\|w_i - w_j\|_2 = 3\sqrt{2}/(4\sqrt{\pi})$ and $\rho < (10000k)^{-8}$. Specifically, we first apply a rotation to $\{B_p''\}_{p=1}^{k}$ such that (105) holds. Then, by (103), we then apply another rotation that fixes the x_1 axis, so that (106) holds. By (102) and (103), each of these two rotations moves a given unit vector in $\mathbb{R}^n$ a distance not more than $12(6k\rho)^{1/8}$. And since we are rotating three polygonal cones with two faces each, (104) holds.

Using (104) and the triangle inequality, $(\sum_{p=1}^{k} \|1_{A_p} - 1_{B_p'}\|_{L_2(\gamma_n)}^2)^{1/2} < 20(6k\rho)^{1/16}$. Also, using that $c_\ell = 0$ for $|\ell| = 1, \ell_1 = 0$, (105) implies that $B_i' \cap B_j' \subseteq \{x \in \mathbb{R}^n : x_1 = 0\}$, and we may assume that $B_i' \subseteq \{x \in \mathbb{R}^n : x_1 \geq 0\}$.

Let $n_i' \in \mathbb{R}^n$ denote the interior unit normal of B_i' such that n_i' is normal to $(\partial B_i') \setminus B_j'$, and let $n_j' \in \mathbb{R}^n$ denote the interior unit normal of B_j' such that n_j' is normal to $(\partial B_j') \setminus B_i'$. Then, define B_i, B_j such that

$$\begin{aligned} B_i &:= B_i' \cup \{x \in \mathbb{R}^n : x_1 \geq 0 \wedge \langle n_i', x/\|x\|_2\rangle \geq -4\rho^{21/20}/\sqrt{-3\log\rho}\}, \\ B_j &:= B_j' \cup \{x \in \mathbb{R}^n : x_1 \leq 0 \wedge \langle n_j', x/\|x\|_2\rangle \geq -4\rho^{21/20}/\sqrt{-3\log\rho}\}. \end{aligned} \tag{107}$$

Since $B_i \cup B_j$ is symmetric with respect to reflection across $B_i \cap B_j = B_i' \cap B_j'$, equation (105) implies that there is a $\lambda > 0$ such that

$$\int_{\mathbb{R}^n} x(1_{A_i}(x) - 1_{A_j}(x)) d\gamma_n(x) = \lambda \int_{\mathbb{R}^n} x(1_{B_i}(x) - 1_{B_j}(x)) d\gamma_n(x). \tag{108}$$

Combining (5), (7), and (108), there exists $|b_1| < 50(6k\rho)^{1/16}$ such that

$$LT_\rho(1_{A_i} - 1_{A_j})(x) - LT_\rho(1_{B_i} - 1_{B_j})(x) - x_1 b_1 =: \sum_{\ell \in \mathbb{N}^n : |\ell| \geq 2} b_\ell |\ell| \rho^{|\ell|-1} h_\ell(x)\sqrt{\ell!}. \tag{109}$$

Choose ρ_1 so that $0 < \rho < \rho_1$ implies $100k^{1/16}\sum_{m=2}^{\infty} m(m+n-1)^n \rho^{m-2} m^n 3^m (-\log\rho^3)^{m/2} < \rho^{-1/80}/20$. Recall that the number of $\ell \in \mathbb{N}^n$ such that $|\ell| = m$ is equal to $\frac{m+n-1!}{m!(n-1)!} \leq (m+n-1)^n$. Note that, $|b_\ell| < 100k^{1/16}\rho^{1/16}$, $\ell \in \mathbb{N}^n$, $|\ell| \geq 2$. By (109), Lemma 4.1, and

since $||x||_2^2 \leq -\log \rho^3$,

$$
\begin{aligned}
&\left|LT_\rho(1_{A_i} - 1_{A_j})(x) - LT_\rho(1_{B_i} - 1_{B_j})(x) - x_1 b_1\right| \\
&\qquad \leq 100 k^{1/8} \rho^{17/16} \sum_{\ell \in \mathbb{N}^n \colon |\ell| \geq 2} |\ell|\, \rho^{|\ell|-2} \left|h_\ell(x)\right| \sqrt{\ell!} \\
&\qquad \leq 100 k^{1/8} \rho^{17/16} \sum_{\ell \in \mathbb{N}^n \colon |\ell| \geq 2} |\ell|\, \rho^{|\ell|-2} \, |\ell|^n \, 3^{|\ell|} \prod_{i=1}^n \max\{1, |x_i|^{\ell_i}\} \\
&\qquad \leq 100 k^{1/8} \rho^{17/16} \sum_{m=2}^\infty m(m+n-1)^n \rho^{m-2} m^n 3^m (-\log \rho^3)^{m/2} \leq \rho^{21/20}/20.
\end{aligned} \tag{110}
$$

From Lemma 3.2 and (7), for $x = (x_1, \ldots, x_n)$, with $B_i \cap B_j \subseteq \{x \in \mathbb{R}^n \colon x_1 = 0\}$,

$$
x \in B_i \cup B_j \,\wedge\, ||x||_2^2 \leq -\log \rho^3 \Longrightarrow \operatorname{sign}(x_1) \cdot LT_\rho(1_{B_i} - 1_{B_j})(x) > (1/10)\, |x_1| \,. \tag{111}
$$

Then (110) and (111) show that

$$
|x_1| > \rho^{21/20} \,\wedge\, ||x||_2^2 \leq -\log \rho^3 \,\wedge\, x \in B_i \cup B_j \Longrightarrow \operatorname{sign}(x_1) \cdot LT_\rho(1_{A_i} - 1_{A_j})(x) > 0. \tag{112}
$$

By (96), Lemma 2.7, and by applying (112) for all $i', j' \in \{1, \ldots, k\}$, $i' \neq j'$, along with the inclusion-exclusion principle,

$$
|x_1| > \rho^{21/20} \,\wedge\, ||x||_2^2 \leq -\log \rho^3 \,\wedge\, x \in B_i' \cup B_j' \Longrightarrow \operatorname{sign}(x_1) \cdot (1_{A_i}(x) - 1_{A_j}(x)) > 0. \tag{113}
$$

From (113),

$$
\begin{gathered}
x \in B_i' \cup B_j' \,\wedge\, |d(x, (\partial B_i') \cup (\partial B_j'))| > \rho^{21/20} \,\wedge\, ||x||_2^2 \leq -\log \rho^3 \\
\Longrightarrow \operatorname{sign}(x_1) \cdot (1_{A_i}(x) - 1_{A_j}(x)) > 0.
\end{gathered} \tag{114}
$$

Recall that there exists a 2-dimensional plane $\Pi \subseteq \mathbb{R}^n$ such that $0 \in \Pi$ and such that, for all $p \in \{1, \ldots, k\}$, $\int_{A_p} x d\gamma_n(x) \in \Pi$. Define

$$
\begin{aligned}
S := \operatorname{span}\bigg\{ &\int_{\mathbb{R}^n} (1_{B_i'}(x) - 1_{B_j'}(x)) x d\gamma_n(x), \int_{\mathbb{R}^n} (1_{B_i'}(x) - 1_{(B_i' \cup B_j')^c}(x)) x d\gamma_n(x) \\
&, \int_{\mathbb{R}^n} (1_{B_j'}(x) - 1_{(B_i' \cup B_j')^c}(x)) x d\gamma_n(x) \bigg\}.
\end{aligned}
$$

Note that S is a 2-dimensional plane and $0 \in S$. By (105), $\int_{\mathbb{R}^n} (1_{B_i'}(x) - 1_{B_j'}(x)) d\gamma_n(x) \in \Pi$. Moreover, since $\{B_i', B_j', (B_i' \cup B_j')^c\}$ is a regular simplicial conical partition,

$$
S = \operatorname{span}\left\{ \int_{B_i'} x d\gamma_n(x), \int_{B_j'} x d\gamma_n(x), \int_{(B_i' \cup B_j')^c} x d\gamma_n(x) \right\}.
$$

From (106), S and Π are 2-dimensional planes that both contain the linearly independent vectors $\int_{(B_i' \cup B_j')^c} x d\gamma_n(x)$ and $\int_{\mathbb{R}^n} (1_{B_i'}(x) - 1_{B_j'}(x)) x d\gamma_n(x)$. We therefore conclude that $S = \Pi$. In particular,

$$
\int_{B_p'} x d\gamma_n(x) \in \Pi, \, \forall\, p \in \{i, j\}. \tag{115}
$$

Let $\rho_0 := \min(\rho_1, \varepsilon_1/6k, e^{-20(n+1)^{10^{12}n^3(n+2)!}})$. Using (114), (105) and (115), we can iteratively apply Lemma 4.4 an infinite number of times. Therefore, $\{A_i\}_{i=1}^k$ must be a regular simplicial conical partition. $\square$

Theorem 5.2 (**Main Theorem**). *Let $n \geq 2, k = 3$. There exists $\rho_0 = \rho_0(n,k) > 0$ such that Conjecture 1 holds for $\rho \in (0, \rho_0)$. Moreover, up to orthogonal transformation, the regular simplicial conical partition uniquely achieves the maximum of* (2) *in Conjecture 1.*

Proof. Choose ρ_0 via Theorem 5.1 and let $0 < \rho < \rho_0$. Let $\{B_i\}_{i=1}^k \subseteq \mathbb{R}^n$ be a regular simplicial conical partition. By Theorem 5.1 and the fact that $\Delta_k^0(\gamma_n) \subseteq \Delta_k(\gamma_n)$,

$$\psi_\rho(1_{B_1}, \ldots, 1_{B_k}) = \sup_{(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)} \psi_\rho(f_1, \ldots, f_k). \tag{116}$$

Let $(f_1, \ldots, f_k) \in \Delta_k^0(\gamma_n)$. By (10), $\sum_{i=1}^k \int f_i T_0 f_i d\gamma_n = k(1/k^2) = 1/k$. By the Fundamental Theorem of Calculus and (116),

$$\sum_{i=1}^k \int f_i T_\rho f_i d\gamma_n = \int_0^\rho \left[\frac{d}{d\alpha}\sum_{i=1}^k \int f_i T_\alpha f_i d\gamma_n\right] d\alpha + \frac{1}{k} = \int_0^\rho \psi_\alpha(f_1, \ldots, f_k) d\alpha + \frac{1}{k}$$

$$\leq \int_0^\rho \psi_\alpha(1_{B_1}, \ldots, 1_{B_k}) d\alpha + \frac{1}{k} = \int_0^\rho \left[\frac{d}{d\alpha}\sum_{i=1}^k \int 1_{B_i} T_\alpha 1_{B_i} d\gamma_n\right] d\alpha + \frac{1}{k} = \sum_{i=1}^k \int 1_{B_i} T_\rho 1_{B_i} d\gamma_n.$$

$\square$

By [9][Theorem 1.10,Theorem 7.4], Theorem 5.2 implies a weak form of the Plurality is Stablest Conjecture. While the following result is quite far from Conjecture 2 and might not be of immediate use to complexity theory, it is included to indicate a possible application of Theorem 5.2. Essentially, if we bypass [9][Theorem 7.1], then Conjecture 2 follows. However, by avoiding [9][Theorem 7.1], we must make very restrictive assumptions on the function f in Conjecture 2. Nevertheless, [9][Theorem 7.4] shows that the class of functions f described in Corollary 5.3 is nonempty.

Note that the most straightforward application of Theorem 5.2 only gives vacuous cases of Conjecture 2, in which $0 < \rho < \rho_0(n,k)$. In particular, since Theorem 5.2 requires $0 < \rho < \rho_0(n,k)$, by (10) we must take $\varepsilon < 3k\rho$ to get a nontrivial statement in Conjecture 2. In this case, [9] gives τ with $\log \tau = -C(\log(\varepsilon))^2(1/\varepsilon)$, so that τ becomes a function of ρ. Since we provide a ρ with inverse exponential dependence on n, then τ also has inverse exponential dependence on n. Thus, no function f can satisfy the assumptions of Conjecture 2 in this case. To avoid this issue, we modify Conjecture 2 as follows.

Corollary 5.3 (Weak Form of Plurality is Stablest). *Let $\rho_0(n,k)$ as given by Theorem 5.2. Fix $n \geq 2$, $k = 3$, and Let $N := \log\log\log\log\log(n) \geq 1$. Let $0 < \rho < \rho_0(N,k) < 1/2$, $\varepsilon > 0$, $\tau = \tau(\varepsilon, k) > 0$. Let $f\colon \{1,\ldots,k\}^n \to \Delta_k$ with $\sum_{\sigma\in\{1,\ldots,k\}^n\colon \sigma_j\neq 0}(\widehat{f_i}(\sigma))^2 \leq \tau$ for all $i \in \{1,\ldots,k\}$, $j \in \{1,\ldots,n\}$. Assume that there exists $0 < m < N$ and $g\colon \mathbb{R}^m \to \Delta_k$ with $\int g d\gamma_m = \frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n} f(\sigma)$, and such that*

$$\left|\int \langle g, T_\rho g\rangle d\gamma_n - \frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n} \langle f(\sigma), T_\rho f(\sigma)\rangle\right| < \varepsilon.$$

Then part (a) of Conjecture 2 holds. From [9][Theorem 7.4], this class of f is nontrivial.

In Corollary 5.3, for $g = (g_1, \ldots, g_k)$ with $g_i \colon \mathbb{R}^m \to [0,1]$, $i \in \{1, \ldots, k\}$, we have used the notation $T_\rho g := (T_\rho g_1, \ldots, T_\rho g_k)$.

Unfortunately, the proof of Theorem 5.2 fails for small negative ρ, as we now show.

Theorem 5.4. *Fix $k = 3$, $n \geq 2$. Define $\Delta_k^0(\gamma_n)$ as in Definition 2.2 and define ψ_ρ as in (14). Let $\{B_i\}_{i=1}^k \subseteq \mathbb{R}^n$ be a regular simplicial conical partition. Then there exists $\rho_2 = \rho_2(n,k) > 0$ such that, for $\rho \in (-\rho_2, 0)$, $(1_{B_1}, \ldots, 1_{B_k})$ does not achieve the following supremum.*

$$\sup_{(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)} \sum_{i=1}^k \int f_i L T_\rho f_i d\gamma_n = \sup_{(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)} \psi_\rho(f_1, \ldots, f_k).$$

Proof. Let $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$. Fix $i,j \in \{1,\ldots,k\}, i \neq j$. Let $\sigma \colon \mathbb{R}^n \to \mathbb{R}^n$ denote reflection across $B_i \cap B_j$. Since $B_i = \sigma(B_j)$, by (16), it suffices to find $i,j \in \{1,\ldots,k\}$ and $x \in B_i$ such that $LT_\rho 1_{B_i}(x) < LT_\rho 1_{B_j}(x)$. By replacing $\{B_i\}_{i=1}^k$ with $\{\tau B_i\}_{i=1}^k$ for $\tau \colon \mathbb{R}^n \to \mathbb{R}^n$ a rotation, we may assume that $\mathrm{span}\{z_i\}_{i=1}^k = \mathrm{span}\{e_1, e_2\}$. Moreover, we may assume $B_i \cap B_j \subseteq \{x \in \mathbb{R}^n \colon x_1 = 0\}$ and $B_i \subseteq \{x \in \mathbb{R}^n \colon x_1 \geq 0\}$. Let $y := (\sqrt{3}/2)e_1 + (1/2)e_2$, $\widetilde{y} := -(1/2)e_1 + (\sqrt{3}/2)e_2$. Fix $x \in B_i$ with $\langle x, \widetilde{y}\rangle > 0$ also fixed. From (21) and the fact that $\rho < 0$, there exists $c = c(\langle x, \widetilde{y}\rangle) > 0$ such that

$$\left\langle x, \frac{1}{\rho}\nabla T_\rho(1_{B_i} - 1_{B_j})(x)\right\rangle = -\langle x, \widetilde{y}\rangle(c + O(e^{-\langle x,y\rangle^2/2})). \tag{117}$$

For $x \in \mathbb{R}^n$ with $\langle x, \widetilde{y}\rangle = 0$, we have, as in Lemma 4.2, and Lemma 3.1,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\left(\sum_{i=1}^n (1-y_i^2)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right)\right| \\ &\qquad\leq 2\left|\int_{B(0,\rho||x||_2)}\sum_{i=1}^n (1-y_i^2)d\gamma_n(y)\right| \leq 200(n+1)!((\rho\,||x||_2)^n + 1)e^{-\rho^2||x||_2^2}.\end{aligned}$$

So, a derivative bound as in the proof of (19) shows

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\left(\sum_{i=1}^n (1-y_i^2)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right)\right| \\ &\qquad\leq \rho\langle x, \widetilde{y}\rangle 200(n+2)! + 200(n+1)!((\rho\,||x||_2)^n + 1)e^{-\rho^2||x||_2^2}.\end{aligned} \tag{118}$$

By (7),

$$\begin{aligned}LT_\rho f(x) &= \frac{1}{\rho}(\langle x, \nabla T_\rho f(x)\rangle - \Delta T_\rho f(x)) \\ &= \langle x, T_\rho(\nabla f)(x)\rangle + \frac{\rho}{1-\rho^2}\int_{\mathbb{R}^n}\left(\sum_{i=1}^n (1-y_i^2)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})\right)d\gamma_n(y).\end{aligned} \tag{119}$$

So, choose $\rho < (c/8)(200(n+2)!)^{-1}$, then choose $\langle x, y\rangle$ sufficiently large, and then combine (117),(118) and (119) with $f = 1_{B_i} - 1_{B_j}$ to get

$$LT_\rho(1_{B_i} - 1_{B_j})(x) < -\langle x, \widetilde{y}\rangle\frac{c}{4}.$$

$\square$

6. Open Problems

There are two problems that are left open in this work. First, Conjecture 1 remains entirely open for $k \geq 4$ partition elements. Some of the results of this work hold for the case $k \geq 4$, and some do not. One of the main issues for the case $k \geq 4$ is that Lemma 2.5 is no longer available. The following conjecture summarizes the main technical issue in proving an analogue of Lemma 2.5 for $k = 4$, $n = 3$. Before we state the conjecture, recall Definition 2.2 and (14).

Conjecture 3. *Let $k = 4$, $n = 3$. Suppose $\{A_i\}_{i=1}^k \subseteq \mathbb{R}^n$ satisfies*

$$\psi_0(1_{A_1}, \ldots, 1_{A_k}) = \sup_{(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)} \psi_0(f_1, \ldots, f_k).$$

Then $\{A_i\}_{i=1}^k$ is a simplicial conical partition.

This result is known to be true if we replace $\Delta_k^0(\gamma_n)$ with $\Delta_k(\gamma_n)$, by [11, 12]. However, the volume constraint of $\Delta_k^0(\gamma_n)$ causes difficulties for the methods of [11, 12].

The second problem that remains open is Conjecture 1 for $\rho < 0$ or for ρ positive and much larger than 0. For ρ with, e.g. $\rho \in (1/2, 1)$, the error bounds that we use in the proof of Theorem 5.2 seem to break down, especially when we apply Lemma 4.4, Lemma 3.2, and (19). So, it seems that our method is not applicable for $\rho \in (1/2, 1)$. However, since the case $\rho \in (1/2, 1)$ relates to geometric multi-bubble problems, whereas the case of small ρ seems to concern entirely different geometric information, it is unclear whether or not a single method could simultaneously solve or interpolate between different values of ρ in Conjecture 1.

Acknowledgements. Thanks to Assaf Naor for guidance and encouragement, and for helpful comments concerning Corollary 5.3. Thanks also to Elchanan Mossel for helpful comments concerning the details of Corollary 5.3 and the vacuous cases of Conjecture 2.

References

[1] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.
[2] P. Austrin. Towards sharp inapproximability for any 2-CSP. *SIAM J. Comput.*, 39(6):2430–2463, 2010.
[3] C. Borell. Geometric bounds on the Ornstein-Uhlenbeck velocity process. *Z. Wahrsch. Verw. Gebiete*, 70(1):1–13, 1985.
[4] M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor. The Grothendieck constant is strictly smaller than Krivine's bound. In *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science—FOCS 2011*, pages 453–462. IEEE Computer Soc., Los Alamitos, CA, 2011.
[5] J. Corneli, I. Corwin, S. Hurder, V. Sesum, Y. Xu, E. Adams, D. Davis, M. Lee, R. Visocchi, and N. Hoffman. Double bubbles in Gauss space and spheres. *Houston J. Math.*, 34(1):181–204, 2008.
[6] A. Frieze and M. Jerrum. Improved approximation algorithms for MAX k-CUT and MAX BISECTION. In *Integer programming and combinatorial optimization (Copenhagen, 1995)*, volume 920 of *Lecture Notes in Comput. Sci.*, pages 1–13. Springer, Berlin, 1995.
[7] L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97(4):1061–1083, 1975.
[8] S. Heilman, A. Jagannath, and A. Naor. Solution of the propeller conjecture in $\mathbb{R}^3$. Preprint, arXiv:1112.2993v1, 2011.
[9] M. Isaksson and E. Mossel. Maximally stable Gaussian partitions with discrete applications. *Israel J. Math.*, 189:347–396, 2012.
[10] S. Khot, G. Kindler, E. Mossel, and R. O'Donnell. Optimal inapproximability results for MAX-CUT and other 2-variable CSPs? *SIAM J. Comput.*, 37(1):319–357, 2007.
[11] S. Khot and A. Naor. Approximate kernel clustering. *Mathematika*, 55(1-2):129–165, 2009.

[12] S. Khot and A. Naor. Sharp kernel clustering algorithms and their associated grothendieck inequalities. Preprint, arXiv:0906.4816v1, 2011.

[13] S. Khot and A. Naor. Grothendieck-type inequalities in combinatorial optimization. *Comm. Pure Appl. Math.*, 65(7):992–1035, 2012.

[14] M. Ledoux. Isoperimetry and Gaussian analysis. In *Lectures on probability theory and statistics (Saint-Flour, 1994)*, volume 1648 of *Lecture Notes in Math.*, pages 165–294. Springer, Berlin, 1996.

[15] E. Mossel, R. O'Donnell, and K. Oleszkiewicz. Noise stability of functions with low influences: invariance and optimality. *Ann. of Math. (2)*, 171(1):295–341, 2010.

[16] P. Raghavendra. Optimal algorithms and inapproximability results for every CSP? [extended abstract]. In *STOC'08*, pages 245–254. ACM, New York, 2008.

[17] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

Courant Institute, New York University, New York NY 10012
E-mail address: heilman@cims.nyu.edu